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Bergshoeff, E.; Roo, M. de

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## $N = 2$ $W$ SUPERGRAVITY

E. BERGSHOEFF\* and M. DE ROO†

*Institute for Theoretical Physics, Nijenborgh 4, 9747 AG Groningen, The Netherlands*

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We quantize the classical gauge theory of  $N = 2$   $w_\infty$  supergravity and show how the underlying  $N = 2$  super- $w_\infty$  algebra gets deformed into an  $N = 2$  super- $W_\infty$  algebra. Both algebras contain the  $N = 2$  super-Virasoro algebra as a subalgebra. We discuss how one can extract from these results information about quantum  $N = 2$   $W_N$  supergravity theories containing a finite number of higher-spin symmetries with superspin  $s \leq N$ . As an example we discuss the case of quantum  $N = 2$   $W_3$  supergravity.

### 1. Introduction

In recent years there has been considerable interest in extensions of the Virasoro algebra containing higher-spin generators. The first example of such a higher-spin extension, the  $W_3$  algebra,<sup>1</sup> contains besides the usual Virasoro spin-2 generator an additional generator of spin-3. Subsequently, a more general set of so-called  $W_N$  algebras, containing higher-spin generators of spin  $2 \leq s \leq N$ , were introduced.<sup>2</sup> Further properties of the  $W_N$  algebras have been discussed in Refs. 3, 4.

One may further generalize the  $W_N$  algebras in different ways. First of all, it is possible to consider  $W$  algebras with an infinite number of higher-spin generators. The first example of such an algebra is the  $w_\infty$  algebra.<sup>5</sup> Other examples are the so-called  $W_\infty$  and  $W_{1+\infty}$  algebras.<sup>6,7</sup>

Secondly, one may consider supersymmetric extensions of the  $W$  algebras, both with a finite as well as with an infinite number of higher-spin generators. The supersymmetric extension of  $w_\infty$  was given in Refs. 8, 9, and of  $W_\infty \oplus W_{1+\infty}$  in Ref. 10. The latter algebra can be defined for arbitrary values of the central charge. It turns out that it is not so easy to construct a similar super- $W$  algebra with a finite number of generators (see, e.g. Refs. 11–20). Most examples of super- $W_N$  algebras<sup>a</sup> given so far exist only for specific values of the central charge. In fact, as far as we know only in two cases the explicit OPE expansions defining a super- $W_N$  algebra

\*Bitnet address: bergshoeff@hgrrug5

†Bitnet address: deroo@hgrrug5

<sup>a</sup>We consider here only quantum algebras. An algebra is called classical with respect to a given field realization if the algebra can be realized as a Poisson bracket algebra between currents which depend on the fields. The algebra is called quantum if, in order to realize the algebra, one needs to make more than single contractions between the currents (the single contractions correspond to the Poisson brackets).

have been given in the literature. These are the  $N = 2$  super- $W_2$  algebra<sup>13,15</sup> and the  $N = 2$  super- $W_3$  algebra.<sup>18</sup> <sup>b</sup> Furthermore, in Ref. 16 the existence of an  $N = 1$  supersymmetric extension of  $W_3$  has been argued.

It has by now become clear that the bosonic higher-spin  $W$  symmetries occur in a number of, quite unexpected, places. To give a few examples, the  $W_N$  symmetries play a role in the context of conformal field theories with  $c \geq 1$ , exceptional modular invariants,<sup>3</sup> nonlinear differential equations, Toda theories<sup>4</sup> and matrix models of 2D gravity.<sup>21</sup> Similarly,  $W_\infty$  symmetries were found in recent studies of the first Hamiltonian structure of the KP hierarchy,<sup>22</sup> matrix models of 2D gravity,<sup>23,24</sup> discrete states,<sup>25</sup> two-dimensional black holes<sup>26</sup> and string field theories.<sup>27</sup>

In most of the above examples the presence of the  $W$  symmetries was discovered *a posteriori*. One could also consider the  $W$  symmetries as fundamental symmetries and treat them on the same footing as the Virasoro symmetries. The aim here is to extend the ordinary string to a so-called " $W$  string." Recently, several steps in this programme have been undertaken, including the gauging of the  $W$  symmetries,<sup>28-31</sup> an investigation of the anomaly-structure in  $W$  gravity,<sup>28,32-39</sup> and a study of the spectrum of  $W$  strings.<sup>40,41</sup>

It seems natural to investigate the role of supersymmetry in the above examples. Supersymmetric  $W$  symmetries were found in, for instance, the super-KP hierarchy.<sup>42-44</sup> Again, one could consider super- $W$  symmetries as fundamental symmetries underlying a  $W$  superstring theory. It is well known from ordinary string theory that the additional supersymmetry brings in attractive features. For instance, it removes the tachyon which is present in the bosonic string spectrum. It is to be expected that similar things will happen in the case of  $W$  superstrings.

With the above motivation in mind we will investigate in this paper the structure of  $W$  supergravity theories. We will define our starting point, which is the classical gauge theory of  $N = 2$   $w_\infty$  supergravity in Sec. 3. The underlying algebra of this gauge theory is discussed in Sec. 2. We will use a representation in which the matter fields are represented by two scalar superfields, corresponding to a two-dimensional target space. Ultimately, our goal is to use a multiscalar representation corresponding to a higher-dimensional target space. Along the lines of the advances which have been made recently in the bosonic case, we will show in Sec. 4 that the theory can be consistently quantized, thereby removing all so-called matter-dependent anomalies. In this process the underlying classical algebra gets deformed into a quantum algebra, as in the bosonic case.<sup>36</sup> We will exhibit the structure of the quantum algebra in Sec. 5 and in Sec. 6 discuss the remaining so-called universal anomalies. Both Secs. 2 and 5, which deal with the classical and quantum algebra, respectively, can be read independently of the rest of the paper.

The main part of this paper deals with the case of  $W_\infty$  symmetries. However, as is well known from the bosonic case,<sup>30</sup> one can sometimes truncate a theory with

<sup>b</sup>The  $N = 2$  super- $W_3$  algebra seems to be the first member of a whole family of quantum  $N = 2$  super- $W_N$  algebras which can be defined for arbitrary values of the central charge.<sup>19,20</sup>

$W_\infty$  symmetries to a theory with  $W_N$  symmetries. We will discuss this point in Sec. 7 and show in which sense our results give information on the structure of  $W_N$  supergravity. In particular, we will discuss the case of  $N = 2$   $W_3$  supergravity. Finally, in Sec. 8 we give our conclusions and in the appendix we give some representative examples of OPE expansions.

We indicate Planck's constant  $\hbar$  explicitly when we want to emphasize the distinction between classical and quantum aspects.

## 2. The $N = 2$ Super- $w_\infty$ Algebra

The  $N = 2$  super- $w_\infty$  algebra<sup>9</sup> is a higher-spin extension of the  $N = 2$  super-Virasoro algebra with generators  $w^{(s)} (s = 1, 3/2, 2, \dots)$ . The algebra can be defined by giving the (singular part of the) OPE expansions of the generators. The OPE expansion of two generators  $w^{(s)}, w^{(t)}$  where both  $s$  and  $t$  are integer is given by (we set  $\hbar = 1$  in this section)

$$w^{(s)}(1)w^{(t)}(2) \sim -2 \frac{\theta_{12} w^{(s+t-1/2)}}{z_{12}}. \quad (1)$$

In all other cases the OPE expansion is given by

$$w^{(s)}(1)w^{(t)}(2) \sim (-)^{|2s+1|_2} \left\{ \left( s+t-\frac{3}{2} \right) \frac{\theta_{12} w^{(s+t-3/2)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 w^{(s+t-3/2)}}{z_{12}} + \left( s-\frac{1}{2} \right) \frac{\theta_{12} \partial_2 w^{(s+t-3/2)}}{z_{12}} \right\}, \quad (2)$$

where  $|s|_2$  is equal to zero for  $s$  even and one for  $s$  odd. Furthermore, we have defined  $z_{12} = z_1 - z_2 + \theta_1 \theta_2$ . The superspace coordinates are  $(Z, \bar{Z}) = (z, \theta, \bar{z}, \bar{\theta})$ . The superspace differential operators  $D, \bar{D}$  are defined by

$$D = \frac{\partial}{\partial \theta} - \theta \partial, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \bar{\partial}, \quad (3)$$

where  $\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$  (corresponding to an Euclidean-signature on the world-sheet),  $\partial_\theta, \partial_{\bar{\theta}}$  are left-derivatives and  $D^2 = -\partial, \bar{D}^2 = -\bar{\partial}$ . Note that  $D_1 z_{12} = D_2 z_{12} = -\theta_{12}$  and  $D_1 \theta_{12} = -D_2 \theta_{12} = 1$ . We will often use the short-hand notation  $w^{(s)}(1)$  to indicate  $w^{(s)}(Z_1, \bar{Z}_1)$ , etc. From Eqs. (1) and (2) one can recover the commutation relations of the generators of the algebra by multiplying the OPE's by the parameters of the corresponding transformations and integrating over the superspace coordinates.

It turns out that it is possible to extend the  $N = 2$  super- $w_\infty$  algebra with an additional  $s = 1/2$  generator  $w^{(1/2)}$  with<sup>c</sup>  $w^{(1/2)}(1)w^{(1/2)}(2) \sim 0$ . The OPE expansion of  $w^{(1/2)}$  with  $w^{(s)}$  ( $s$  integer) is given by

<sup>c</sup>In cases where we would like to stress the presence of the  $s = 1/2$  generator we will call the extended algebra, in analogy with the terminology  $w_\infty$  versus  $w_{1+\infty}$ , the  $N = 2$  super- $w_{1/2+\infty}$  algebra.

$$w^{(1/2)}(1)w^{(s)}(2) \sim \frac{w^{(s-1/2)}}{z_{12}} + \left\{ (s-1) \frac{\theta_{12}w^{(s-1)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 w^{(s-1)}}{z_{12}} \right\}. \quad (4)$$

For half-integer  $s$  ( $s \geq 3/2$ ), the OPE expansion is given by

$$\begin{aligned} w^{(1/2)}(1)w^{(s)}(2) \sim & \left\{ (s-1) \frac{\theta_{12}w^{(s-1)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 w^{(s-1)}}{z_{12}} \right\} \\ & + \left\{ \frac{1}{2} \left( s - \frac{3}{2} \right) \frac{w^{(s-3/2)}}{z_{12}^2} - \frac{1}{4} \frac{\theta_{12} D_2 w^{(s-3/2)}}{z_{12}^2} \right. \\ & \left. - \frac{1}{4} \frac{\partial_2 w^{(s-3/2)}}{z_{12}} \right\}. \end{aligned} \quad (5)$$

The  $N = 2$  super- $w_\infty$  algebra contains an  $N = 2$  super-Virasoro subalgebra which is generated by  $\{w^{(1)}, w^{(3/2)}\}$ :

$$\begin{aligned} w^{(1)}(1)w^{(1)}(2) & \sim -2 \frac{\theta_{12}w^{(3/2)}}{z_{12}}, \\ w^{(3/2)}(1)w^{(1)}(2) & \sim \left\{ \frac{\theta_{12}w^{(1)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 w^{(1)}}{z_{12}} + \frac{\theta_{12}\partial_2 w^{(1)}}{z_{12}} \right\}, \\ w^{(3/2)}(1)w^{(3/2)}(2) & \sim \left\{ \frac{3}{2} \frac{\theta_{12}w^{(3/2)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 w^{(3/2)}}{z_{12}} + \frac{\theta_{12}\partial_2 w^{(3/2)}}{z_{12}} \right\}. \end{aligned} \quad (6)$$

The superfields  $w^{(1/2)}$  and  $\{w^{(s)}, w^{(s+1/2)}\}$  with  $s$  integer form  $N = 2$  multiplets with respect to the  $osp(2, 2)$  subalgebra of the  $N = 2$  super-Virasoro subalgebra. Here  $w^{(1/2)}$  constitutes a so-called  $N = 2$  scalar multiplet. The  $osp(2, 2)$  subalgebra is defined by the  $s = 1, 3/2$  transformations where the parameters  $k_{(1)}, k_{(3/2)}$  which multiply the currents  $w^{(1)}, w^{(3/2)}$  satisfy the conditions

$$D^3 k_{(1)} = D^5 k_{(3/2)} = 0. \quad (7)$$

It is possible to perform different truncations of the  $N = 2$  super- $w_\infty$  algebra. We first consider the ones that maintain the  $N = 2$  supersymmetry. It turns out that, for a given positive integer  $M \geq 1$ , it is consistent to retain only the  $N = 2$  multiplets  $\{w^{(s)}, w^{(s+1/2)}\}$  ( $s$  integer) with  $s = 1 + kM$ ,  $k = 0, 1, 2, \dots$ . We denote this algebra by  $N = 2$  super- $w_{\infty/M}$ . A similar set of truncations has been discussed in the bosonic case in the second reference of Ref. 28. For  $M = 1$  we recover the original  $N = 2$  super- $w_\infty$  algebra. Only for  $M = 1$ , it is possible to extend the algebra by an  $s = 1/2$  generator to an  $N = 2$  super- $w_{1/2+\infty}$  algebra as indicated above.

We next consider truncations giving algebras with  $N = 1$  supersymmetry.<sup>8,9</sup> One possibility is to retain only the generators  $w^{(s)}$  with half-integer  $s$ . One can then further truncate the algebra by keeping only the generators  $s = 3/2 + kM$  with  $k = 0, 1, 2, \dots$  and  $M \geq 1$  a given integer. Another possibility is to keep only

the generators  $w^{(s)}$  with  $s$  even or  $s + 1/2$  even. We will denote the latter algebra by  $N = 1$  super- $w_\infty$ .

It is also possible to truncate the  $N = 2$  super- $w_\infty$  algebra to a finite set of generators by keeping only the  $N = 2$  multiplets  $\{w^{(s)}, w^{(s+1/2)}\}$  with  $s \leq M$  for a given integer  $M$ , giving an  $N = 2$  super- $w_M$  algebra. The OPE's of the  $N = 2$  super- $w_M$  algebra are given by those of the  $N = 2$  super- $w_\infty$  algebra with the restriction that

$$w^{(s)}(1)w^{(t)}(2) \sim 0, \tag{8}$$

for  $s + t - 1/2 > M$  if  $s, t$  integer and  $s + t - 3/2 > M$  in all other cases. This generalizes a similar truncation that takes place in the bosonic case.<sup>45,46</sup> For clarity we give the field content of some of the truncated algebras in the table below.

Table 1.  $N = 1$  superfield content of some classical super- $w$  algebras. Each spin- $s$  superfield contains two components with spin  $(s, s + 1/2)$ .

Superalgebra	$N = 1$ field content
$N = 2$ super- $w_{1/2+\infty}$	$1/2, 1, 3/2, 2, \dots$
$N = 2$ super- $w_\infty$	$1, 3/2, 2, 5/2, \dots$
$N = 2$ super- $w_{\infty/2}$	$1, 3/2, 3, 7/2, \dots$
$N = 1$ super- $w_\infty$	$3/2, 2, 7/2, 4, \dots$
$N = 2$ super- $w_M$	$1/2, 1, 3/2, 2, \dots, M, M + 1/2$

Finally, we note that the bosonic subalgebra of  $N = 2$  super- $w_{1/2+\infty}$  ( $N = 2$  super- $w_\infty$ ) is given by the direct sum  $w_{1+\infty} \oplus w_{1+\infty}$  ( $w_\infty \oplus w_{1+\infty}$ ).

### 3. Classical $N = 2$ $w_\infty$ -supergravity

The classical theory of chiral  $N = 2$   $w_\infty$ -supergravity<sup>d</sup> that will form our starting point is described by the action  $S = 1/\pi \int d^2Z L$ , where  $L$  is given by<sup>50 e</sup>

$$L = D\phi\bar{D}\bar{\phi} + \sum_{s=1/2}^{\infty} A_{(s)}w^{(s)}. \tag{9}$$

The matter is described by two real scalar superfields  $\phi, \bar{\phi}$ . The currents  $w^{(s)}$  ( $s = 1/2, 1, 3/2, 2, \dots$ ) depend on the matter fields  $\phi, \bar{\phi}$ . Taking single contractions

<sup>d</sup>For results on the gauging of  $W_3$ -supergravity theories, see Refs. 28, 47, 48. For the gauging of a super- $W_\infty$  algebra, see Ref. 49. It is interesting in its own right to compare the quantum theory of the  $N = 2$   $w_\infty$ -supergravity theory we consider in this paper with the quantum theory of the  $N = 2$   $W_\infty$ -supergravity theory of Ref. 49. It is not obvious to us what the exact relationship between the two quantum theories is.

<sup>e</sup>Note that we have given here the kinetic term in an off-diagonal basis. After diagonalization one ends up with the kinetic terms for two scalar fields with a relative minus sign. One might therefore consider these two scalars as the coordinates of a superstring moving in a  $d = 2$  target space with Lorentzian signature.

between these currents (or, equivalently, taking Poisson brackets) one finds the OPE's corresponding to the  $N = 2$  super- $w_\infty$  algebra given in the previous section. Explicitly, the currents are given by

$$\begin{aligned} w^{(s)} &= (\partial\phi)^{s-1} D\phi D\bar{\phi} & (s \text{ integer}), \\ w^{(s)} &= (\partial\phi)^{s-1/2} D\bar{\phi} + \frac{1}{2} D\{D\phi(\partial\phi)^{s-3/2} D\bar{\phi}\} & (s \text{ half-integer}). \end{aligned} \quad (10)$$

We have also introduced gauge fields  $A_{(s)}$ . We note that  $A_{(s)}, w^{(s)}$  are commuting (anticommuting) for integer (half-integer)  $s$ . The two-point function of  $\phi, \bar{\phi}$  is given by

$$\langle \phi(Z_1, \bar{Z}_1) \bar{\phi}(Z_2, \bar{Z}_2) \rangle = -\hbar \ln z_{12} \bar{z}_{12}. \quad (11)$$

The action is invariant under  $N = 2$   $w_\infty$  transformations. Under a spin- $s$  transformation with  $(Z, \bar{Z})$ -dependent parameter  $k_{(s)}$  the  $\phi$  matter field undergoes the following transformation:<sup>f</sup>

$$\delta(k_{(s)})\phi(2) = \hbar^{-1} \sum_{s \geq 1/2} \oint \frac{dZ_1}{2\pi i} k_{(s)}(1) w^{(s)}(1) \phi(2) \quad (12)$$

and similarly for  $\bar{\phi}$ . Note that  $k_{(s)}$  is commuting (anticommuting) for half-integer (integer)  $s$ . Using the explicit form of the currents  $w^{(s)}$  and the useful formula

$$\phi(2) = \frac{1}{2\pi i} \oint dZ_1 \frac{\theta_{12}}{z_{12}} \phi(1), \quad (13)$$

we find

$$\begin{aligned} \delta\phi &= \sum_{s=1/2, 3/2, \dots}^{\infty} \left\{ k_{(s)} (\partial\phi)^{s-1/2} - \frac{1}{2} Dk_{(s)} D\phi (\partial\phi)^{s-3/2} \right\} \\ &\quad + \sum_{s=1, 2, \dots}^{\infty} k_{(s)} D\phi (\partial\phi)^{s-1}, \\ \delta\bar{\phi} &= \sum_{s=1/2, 3/2, \dots}^{\infty} \left\{ -\left(s - \frac{1}{2}\right) D(k_{(s)} (\partial\phi)^{s-3/2} D\bar{\phi}) \right. \\ &\quad \left. + \frac{1}{2} Dk_{(s)} (\partial\phi)^{s-3/2} D\bar{\phi} + \frac{1}{2} \left(s - \frac{3}{2}\right) D(Dk_{(s)} D\phi (\partial\phi)^{s-5/2} D\bar{\phi}) \right\} \\ &\quad + \sum_{s=1, 2, \dots}^{\infty} \{-k_{(s)} (\partial\phi)^{s-1} D\bar{\phi} - (s-1) D(k_{(s)} D\phi (\partial\phi)^{s-2} D\bar{\phi})\}. \end{aligned} \quad (14)$$

<sup>f</sup>We use a notation where the one-dimensional (anticommuting) integration measure is indicated by  $dZ$ . The two-dimensional (commuting) integration measure is denoted by  $d^2Z \equiv dZ d\bar{Z}$ .

We note that the variation of the kinetic term in the action cancels against the inhomogeneous variation  $\bar{D}k_{(s)}$  of the gauge fields, i.e.

$$\int d^2 Z \left( \delta \{ D\phi \bar{D}\bar{\phi} \} + \sum_{s \geq 1/2} \bar{D}k_{(s)} w^{(s)} \right) = 0. \quad (15)$$

The remaining variation  $\hat{\delta}A_{(s)}$  of the gauge fields, defined by  $\delta A_{(s)} \equiv \bar{D}k_{(s)} + \hat{\delta}A_{(s)}$ , can be determined by the requirement that

$$\int d^2 Z \sum_{s \geq 1/2} \{ \hat{\delta}A_{(s)} w^{(s)} + A_{(s)} \delta w^{(s)} \} = 0, \quad (16)$$

where  $\delta w^{(s)}$  may be calculated by applying the general formula (12) and the OPE expansions of the  $N = 2$  super- $w_\infty$  algebra. We thus obtain the following expression for the transformation rules of the gauge fields under the spin  $s \geq 1$  transformations:

$$\delta A_{(s)} = \bar{D}k_{(s)} - 2 \sum_{t=1,2,\dots}^{s-1/2} A_{(t)} k_{(s-t+1/2)} \quad (17)$$

for half-integer  $s$  and integer  $t$  and

$$\begin{aligned} \delta A_{(s)} = \bar{D}k_{(s)} + \sum_{t=\frac{1}{2},1,\dots} \left\{ - \left( t - \frac{1}{2} \right) A_{(t)} \partial k_{(s-t+3/2)} \right. \\ \left. + \frac{1}{2} (-)^{|2t|_2} D A_{(t)} D k_{(s-t+3/2)} + (s-t+1) \partial A_{(t)} k_{(s-t+3/2)} \right\} \end{aligned} \quad (18)$$

in all other cases. It is understood that  $k_{(s)} \equiv 0$  if  $s \leq 1/2$ . Under the spin  $s = 1/2$  transformations the gauge fields transform as follows:

$$\begin{aligned} \delta A_{(s)} = - \left( s + \frac{1}{2} \right) A_{(s+1)} \partial k_{(1/2)} + \frac{1}{2} D A_{(s+1)} D k_{(1/2)} \\ - \frac{1}{2} (s+1) A_{(s+3/2)} \partial D k_{(1/2)} + \frac{1}{4} D A_{(s+3/2)} \partial k_{(1/2)} \\ - \frac{1}{4} \partial A_{(s+3/2)} D k_{(1/2)} \end{aligned} \quad (19)$$

for integer  $s$ . For half-integer  $s$  we have

$$\delta A_{(s)} = - A_{(s+1/2)} D k_{(1/2)} - \left( s + \frac{1}{2} \right) A_{(s+1)} \partial k_{(1/2)} - \frac{1}{2} D A_{(s+1)} D k_{(1/2)}. \quad (20)$$

On both the matter fields  $\phi, \bar{\phi}$  as well as on the gauge fields  $A_{(s)}$  the commutator algebra of the  $k_{(s)}$  transformations closes and corresponds to the classical  $N = 2$  super- $w_\infty$  algebra.



We should comment on the gauging of the lowest-spin  $s = 1/2$  transformation. At first sight one might be surprised that it is possible to gauge this transformation. Indeed, in the one-scalar realization of the bosonic  $w_\infty$ -gravity theory the lowest-spin  $s = 1$  transformation is not gauged.<sup>30</sup> The reason for this difference is that we are working here with a two-scalar realization where the lowest-spin  $s = 1/2$  current is given by  $w^{(1/2)} = D\bar{\phi}$  and, since  $\langle\bar{\phi}(1)\bar{\phi}(2)\rangle = 0$ , a single contraction between two  $s = 1/2$  currents does not give a central term. Therefore, one can treat the  $s = 1/2$  transformation on the same footing as the higher-spin transformations. On the other hand, in the usual one-scalar formulation of the bosonic  $w_\infty$ -gravity theory the lowest-spin  $s = 1$  current is given by  $w^{(1)} = \partial\phi$  and, since now  $\langle\phi(1)\phi(2)\rangle \neq 0$ , a single contraction between two  $s = 1$  currents yields a central term, giving a “classical anomaly.”<sup>8</sup>

It is instructive to take the bosonic limit of the  $N = 2$   $w_\infty$ -supergravity theory. In this limit we are left with two real scalars  $\phi, \bar{\phi}$  with the two-point function given by

$$\langle\partial\phi(1)\partial\bar{\phi}(2)\rangle = \frac{\hbar}{(z_1 - z_2)^2}. \quad (21)$$

The Lagrangian for these scalars reads

$$L = \partial\phi\bar{\partial}\bar{\phi} + \sum_{s=1}^{\infty} A_{(s)} w^{(s)} \quad (22)$$

with the currents  $w^{(s)}$  ( $s = 1, 2, \dots$ ) given by

$$w^{(s)} = (\partial\phi)^{s-1} \partial\bar{\phi}. \quad (23)$$

The corresponding Poisson bracket algebra is the bosonic  $w_{1+\infty}$  algebra. In terms of (the singular part of) the operator product (OPE) expansion of the currents this algebra is given by

$$\hbar^{-1} w^{(s)}(1) w^{(t)}(2) \sim (s+t-2) \frac{w^{(s+t-2)}}{(z_1 - z_2)^2} + (s-1) \frac{\partial_2 w^{(s+t-2)}}{z_1 - z_2}. \quad (24)$$

Note that we have ended up with a two-scalar realization of  $w_{1+\infty}$ -gravity. The  $s = 1$  generator can be truncated consistently, leading to a two-scalar realization of  $w_\infty$ -gravity. In Ref. 30 a one-scalar realization of  $w_\infty$  gravity was obtained.

It is interesting to compare the two-scalar realization of the  $w_\infty$  and  $w_{1+\infty}$  algebras we just found with the two-scalar realization found recently in Ref. 51. It turns out that there is a whole one-parameter family of two-scalar realizations of  $w_\infty$  with currents given by

$$w^{(s)} = (\partial\phi)^{s-1} \partial\bar{\phi} + \alpha \frac{s-2}{s} (\partial\phi)^s. \quad (25)$$

<sup>8</sup>The gauging of algebras with central charges has been discussed in Ref. 52.

Indeed, one may verify that for any choice of the parameter  $\alpha$  the Poisson-bracket algebra of the above currents is equal to  $w_\infty$ . The two-scalar realization of Ref. 51 corresponds to the choice  $\alpha = 1$ . We find that only for  $\alpha = 0$  a single contraction between two  $s = 1$  currents does not yield a central term. Therefore, only for  $\alpha = 0$  does the inclusion of the  $s = 1$  current lead to a  $w_{1+\infty}$  algebra without central extension.

The above realizations of the classical  $w_\infty$  algebra cannot be extended to a realization at the quantum level (i.e. taking multiple contractions between the currents) for arbitrary  $\alpha$ . From Ref. 51 it is clear that, without introducing any further fields, this is possible for  $\alpha = 1$ . In our case, with  $\alpha = 0$  and the  $s = 1$  generator included, it is also possible but we have to modify the currents with terms bilinear in fermions. These are of course exactly the fermions which occur in the  $N = 2$   $w_\infty$ -supergravity theory.<sup>h</sup>

#### 4. Quantization

We now proceed to quantize the chiral  $N = 2$   $w_\infty$ -supergravity theory. In this section we will closely follow a similar analysis for the bosonic  $w_\infty$ -gravity theory<sup>36</sup> and the bosonic  $w_3$ -gravity theory.<sup>34,35</sup> Like in the bosonic case we should distinguish between matter-dependent anomalies and universal anomalies. The first are generated by supergraphs with external matter fields and are typical for nonlinearly realized symmetries. The latter correspond to supergraphs with external gauge fields only. In this section we will show how the matter-dependent anomalies can be eliminated from the theory by suitable finite renormalizations of the supercurrents and the transformation rules. The universal anomalies will be discussed in Sec. 6.

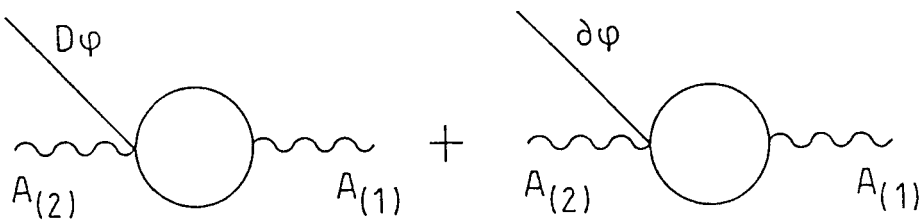


Fig. 1. Two supergraphs giving rise to matter-dependent anomalies.

As an example of a supergraph that can generate matter-dependent anomalies we consider the sample diagrams given in Fig. 1. These are the only two diagrams that have an external  $A_{(1)}$  and  $A_{(2)}$  gauge field and one additional external matter field. The two diagrams can be calculated by evaluating the double contractions in the operator product expansions of  $\int d^2 Z_1 A_{(1)}(Z_1) w^{(1)}(Z_1)$  times  $\int d^2 Z_2 A_{(2)}(Z_2) w^{(2)}(Z_2)$ . The resulting contribution to the effective action is

<sup>h</sup>In the supersymmetric case one is furthermore forced to consider the direct sum  $w_{1+\infty} \oplus w_{1+\infty}$ .

$$\begin{aligned}
\Gamma_{12\phi} &= \frac{\hbar}{\pi^2} \int d^2 Z_1 d^2 Z_2 \left\{ A_1(Z_1) A_2(Z_2) \frac{1}{z_{12}^2} \partial_2 \phi + A_1(Z_1) A_2(Z_2) \frac{\theta_{12}}{z_{12}^3} D_2 \phi \right\} \\
&= \frac{\hbar}{\pi} \int d^2 Z_1 d^2 Z_2 A_1(Z_1) A_2(Z_2) \\
&\quad \times \left\{ -D_1 \frac{\partial_1}{\bar{D}_1} \Delta(Z_1 - Z_2) \partial_2 \phi + \frac{1}{2} \frac{\partial_1^2}{\bar{D}_1} \Delta(Z_1 - Z_2) D_2 \phi \right\} \\
&= -\frac{\hbar}{\pi} \int d^2 Z \left\{ \left( \frac{1}{\bar{D}} \partial D A_1 \right) A_2 \partial \phi + \frac{1}{2} \left( \frac{\partial^2}{\bar{D}} A_1 \right) A_2 D \phi \right\}. \tag{26}
\end{aligned}$$

The delta function  $\Delta(Z_1 - Z_2)$  is defined by

$$\Delta(Z_1 - Z_2) = \delta(z_1 - z_2)(\theta_1 - \theta_2)(\bar{\theta}_1 - \bar{\theta}_2). \tag{27}$$

We have furthermore defined a regularization where

$$\frac{\theta_{12}}{z_{12}} = \pi \frac{1}{\bar{D}_1} \Delta(Z_1 - Z_2). \tag{28}$$

The inverse operator  $1/\bar{D}$  is defined by the relations

$$\frac{1}{\bar{D}} \bar{D} = \bar{D} \frac{1}{\bar{D}} = 1. \tag{29}$$

By taking repeated derivatives one can derive the general identities

$$\begin{aligned}
\frac{\theta_{12}}{z_{12}^n} &= \pi \frac{(-)^{n-1}}{(n-1)!} \frac{\partial_1^{n-1}}{\bar{D}_1} \Delta(Z_1 - Z_2), \\
\frac{1}{z_{12}^n} &= \pi \frac{(-)^{n-1}}{(n-1)!} D_1 \frac{\partial_1^{n-1}}{\bar{D}_1} \Delta(Z_1 - Z_2).
\end{aligned} \tag{30}$$

Under the leading order inhomogeneous terms in the gauge transformations  $\delta A_{(1)} = \bar{D}k_{(1)} + \dots$ ,  $\delta A_{(2)} = \bar{D}k_{(2)} + \dots$  the anomalous variation of  $\Gamma_{12\phi}$  is

$$\begin{aligned}
\delta \Gamma_{12\phi} &= \frac{\hbar}{\pi} \int d^2 Z \left\{ (\partial D k_{(1)}) A_{(2)} \partial \phi - \frac{1}{2} (\partial^2 k_{(1)}) A_{(2)} D \phi \right. \\
&\quad \left. + (\partial D A_{(1)}) k_{(2)} \partial \phi - \frac{1}{2} (\partial^2 A_{(1)}) k_{(2)} D \phi \right\} \\
&\quad + \text{equation of motion terms}. \tag{31}
\end{aligned}$$

It turns out that the anomalous variation (31) can be canceled by adding the finite local counter terms  $L_{1/2} + L_1$ , given by

$$\begin{aligned}
 L_{1/2} = & \sqrt{\hbar} \left( A_{(1)} \partial \phi + A_{(2)} \left( \frac{1}{3} \partial D \phi D \bar{\phi} + \frac{1}{2} (\partial \phi)^2 + \frac{1}{3} \partial \phi \partial \bar{\phi} - \frac{1}{3} D \phi \partial D \bar{\phi} \right) \right. \\
 & + A_{(5/2)} \left( \frac{1}{6} \partial^2 \phi D \bar{\phi} + \frac{1}{3} \partial D \phi \partial \bar{\phi} \right. \\
 & \left. \left. + \frac{1}{2} \partial D \phi \partial \phi - \frac{1}{3} \partial \phi \partial D \bar{\phi} - \frac{1}{6} D \phi \partial^2 \bar{\phi} \right) \right), \\
 L_1 = & \hbar \left( \frac{1}{6} A_{(2)} \partial^2 \phi + \frac{1}{12} A_{(5/2)} \partial^2 D \phi \right)
 \end{aligned} \tag{32}$$

and by simultaneously correcting the transformations of the matter fields  $\phi$  and  $\bar{\phi}$  as well as the gauge field  $A_{(1)}$  by extra terms given by

$$\begin{aligned}
 \delta_{1/2} \phi &= \sqrt{\hbar} \left\{ \frac{1}{3} k_{(2)} \partial D \phi - \frac{1}{3} D(k_{(2)} \partial \phi) + \frac{1}{3} \partial(k_{(2)} D \phi) \right\}, \\
 \delta_{1/2} \bar{\phi} &= \sqrt{\hbar} \left\{ -D k_{(1)} + \frac{1}{3} k_{(2)} \partial D \bar{\phi} - \frac{1}{3} D(k_{(2)} \partial \bar{\phi}) \right. \\
 &\quad \left. - D(k_{(2)} \partial \phi) + \frac{1}{3} \partial(k_{(2)} D \bar{\phi}) \right\}, \\
 \delta_{1/2} A_{(1)} &= -\frac{2}{3} \sqrt{\hbar} \{ A_{(2)} (\partial D k_{(1)}) + (\partial D A_{(1)}) k_{(2)} \}, \\
 \delta_1 \bar{\phi} &= \frac{\hbar}{6} \partial D k_{(2)}.
 \end{aligned} \tag{33}$$

Note that the powers of  $\hbar$  are in agreement with the fact that in two dimensions  $\phi$  and  $\bar{\phi}$  have the dimension of  $\sqrt{\hbar}$ . In varying the effective action the terms of order  $\sqrt{\hbar}$  cancel identically:

$$\delta_0 L_{1/2} + \delta_{1/2} L_0 \equiv 0, \tag{34}$$

where  $L_0$  is the  $\hbar$ -independent part of the Lagrangian given in Eq. (9). The terms of order  $\hbar$  are such that they cancel the anomalous variation (31). They arise in the pattern

$$\delta_0 L_1 + \delta_{1/2} L_{1/2} + \delta_1 L_0. \tag{35}$$

The occurrence of the counterterms (32) implies that the original classical currents  $w^{(1)}$ ,  $w^{(2)}$  and  $w^{(5/2)}$  have received corrections. A similar correction is found to the  $w^{(3/2)}$  current if one considers a Feynman diagram with an external  $A_{(3/2)}$  and  $A_{(2)}$  gauge field and one external matter field. At this point one has found the complete corrections to all currents  $w^{(s)}$  up to and including  $s = 5/2$ . Dimension counting shows that no higher order in  $\hbar$  corrections to these currents are to be expected. The final expressions for the quantum currents  $W^{(s)}$  ( $1/2 \leq s \leq 5/2$ ) are given by

$$\begin{aligned}
W^{(1/2)} &= D\bar{\phi}, \\
W^{(1)} &= D\phi D\bar{\phi} + \sqrt{\hbar}\partial\phi, \\
W^{(3/2)} &= \frac{1}{2}\partial\phi D\bar{\phi} + \frac{1}{2}D\phi\partial\bar{\phi} + \frac{1}{2}\sqrt{\hbar}\partial D\phi, \\
W^{(2)} &= \partial\phi D\phi D\bar{\phi} + \sqrt{\hbar}\left\{\frac{1}{3}\partial D\phi D\bar{\phi} + \frac{1}{2}(\partial\phi)^2\right. \\
&\quad \left. + \frac{1}{3}\partial\phi\partial\bar{\phi} - \frac{1}{3}D\phi\partial D\bar{\phi}\right\} + \frac{\hbar}{6}\partial^2\phi, \\
W^{(5/2)} &= \frac{1}{2}(\partial\phi)^2 D\bar{\phi} - \frac{1}{2}D\phi\partial D\phi D\bar{\phi} + \frac{1}{2}D\phi\partial\phi\partial\bar{\phi} \\
&\quad + \sqrt{\hbar}\left\{\frac{1}{6}\partial^2\phi D\bar{\phi} + \frac{1}{3}\partial D\phi\partial\bar{\phi} + \frac{1}{2}\partial D\phi\partial\phi\right. \\
&\quad \left.- \frac{1}{3}\partial\phi\partial D\bar{\phi} - \frac{1}{6}D\phi\partial^2\bar{\phi}\right\} + \frac{\hbar}{12}\partial^2 D\phi.
\end{aligned} \tag{36}$$

We note that there is an arbitrariness in the above expressions corresponding to the freedom to make redefinitions of the form  $W^{(s)} \rightarrow W^{(s)} + DW^{(s-1/2)} + \dots$ . This arbitrariness can be removed by requiring that the currents transform covariantly under the  $osp(2,2)$  subalgebra of the  $N = 2$  super-Virasoro algebra. This leaves us still with a free undetermined parameter  $\lambda$ .<sup>i</sup> Different choices of this parameter correspond to choosing a different basis of the quantum algebra. For simplicity, we have given the expressions above only in the  $\lambda = 0$  basis. It is straightforward to derive the expressions for arbitrary value of  $\lambda$ . Results for arbitrary  $\lambda$  will be given in the next section.

The transformation rules for the matter fields  $\phi$ , including the corrections (33), follow from the standard expression (12) where one should use now the quantum currents  $W^{(s)}$  instead of the classical currents  $w^{(s)}$ . Finally, the modified transformation rule for the  $A_{(1)}$  gauge field follows from the fact that the OPE expansion of the quantum currents  $W^{(s)}$  differs from that of the classical currents. For instance,

$$w^{(1)}(1)w^{(2)}(2) \sim -2\frac{\theta_{12}w^{5/2}}{z_{12}} \tag{37}$$

but

$$W^{(1)}(1)W^{(2)}(2) \sim -2\frac{\theta_{12}W^{(5/2)}}{z_{12}} + \frac{2}{3}\sqrt{\hbar}\frac{W^{(1)}}{z_{12}^2}. \tag{38}$$

The modified transformation rule of the  $A_{(1)}$  gauge field can now be determined from the requirement that<sup>39</sup>

<sup>i</sup>The situation is different in the one-scalar realization of the bosonic  $w_\infty$ -gravity theory. In order to avoid the introduction of the anomalous  $s = 1$  transformation one must go to a basis with  $\lambda = 0$ .<sup>36</sup>

$$\int d^2 Z \sum_{s \geq 1/2} \{ \hat{\delta} A_{(s)} W^{(s)} + A_{(s)} \hat{\delta} W^{(s)} \} = 0, \quad (39)$$

where  $\hat{\delta} A_{(s)} = \delta A_{(s)} - \bar{D} k_{(s)}$  and<sup>j</sup>

$$\hat{\delta} W^{(t)}(2) = \hbar^{-1} \sum_{s \geq 1/2} \oint \frac{dZ_1}{2\pi i} k_{(s)}(1) W^{(s)}(1) W^{(t)}(2). \quad (40)$$

One can now in principle proceed, by looking at higher-order diagrams with higher-spin external gauge fields, to determine the appropriate modifications to the higher-spin currents  $w^{(s)}$  with  $s \geq 3$  that are needed in order to remove the matter-dependent anomalies. The quantum-corrected currents are denoted by  $W^{(s)}$ . The local part of the effective action is now given by

$$S_{\text{eff}}(\text{local}) = \frac{1}{\pi} \int d^2 Z \{ D\phi \bar{D}\bar{\phi} + \sum_{s \geq 1/2} A_{(s)} W^{(s)} \}. \quad (41)$$

At the same time, the transformation rules of the matter and gauge fields will require higher-spin modifications too. As in the sample diagrams studied above, the modifications to the  $\phi$  and  $\bar{\phi}$  variation will be precisely those that follow by substituting the quantum currents into (12). The modifications to the gauge field variations follow from (39). These constructions can be carried out to arbitrary order in  $\hbar$ .

We will now argue that, like in the bosonic case, the modifications to the Lagrangian and transformation rules can all be understood as a renormalization of the classical  $N = 2$  super- $w_\infty$  algebra to the quantum  $N = 2$  super- $W_\infty$  algebra. First of all, since the modifications to the currents generate the modifications to the matter fields as in (12) it follows that all variations

$$(\delta_0 + \delta_{1/2} + \delta_1 + \dots) D\phi \bar{D}\bar{\phi} \quad (42)$$

of the kinetic term of the effective action are canceled by the variation

$$\sum_{s \geq 1/2} ((\delta - \hat{\delta}) A_{(s)}) W^{(s)} = \sum_{s \geq 1/2} \bar{D} k_{(s)} W^{(s)}. \quad (43)$$

The remaining variation of the effective action is calculated as follows. The variation of the currents  $W^{(s)}$  in the  $A_{(s)} W^{(s)}$  terms in (41) is given by

$$\delta W^{(t)} = \hat{\delta} W^{(t)} - (\text{double} + \text{more contractions}) \quad (44)$$

with  $\hat{\delta} W^{(t)}$  defined in (40). The second term at the right-hand side indicates that to calculate the variation of quantum currents one should only consider the

<sup>j</sup>It is understood here that in the OPE expansion  $W^{(s)}(1)W^{(t)}(2)$  no central charge terms are included. These central charge terms will play a role in Sec. 6 where we will discuss the universal anomalies.

contribution of the single contractions to the OPE expansion  $W^{(s)}(1)W^{(t)}(2)$  in (40). Now using (39) we find that the total variation of  $S_{\text{eff}}(\text{local})$  is given by

$$\delta S_{\text{eff}}(\text{local}) = -\frac{1}{\pi} \int d^2 Z \sum_{s \geq 1/2} A_{(s)} (\hat{\delta} W^{(s)} - (\text{single contractions})). \quad (45)$$

One may verify that this expression is exactly the same (with an opposite sign) to the contribution that follows from the Feynman diagram calculation. Each double or more contraction in the calculation of the quantum algebra corresponds to a particular Feynman diagram. In other words, in the variation of the effective action, the contribution of the local part corresponds to the single contractions in the quantum algebra whereas the contribution from the nonlocal part, arising from the Feynman diagram calculations, corresponds to the double + more contractions. Together they lead to a closed quantum algebra and via (39) to an invariant effective action. Therefore, by construction the cancellation of the matter-dependent anomalies is equivalent to the construction of a closed quantum algebra.

Before proceeding with the cancellation of the universal anomalies in Sec. 6, we will first consider in the next section some basic properties of the quantum  $N = 2$  super- $W_\infty$  algebra. We will describe the algebra in terms of a one-parameter family of bases with parameter  $\lambda$ .<sup>53</sup> To compare with the results of this section one should take the basis corresponding to  $\lambda = 0$ .

In particular, we will give in the next section a closed expression for the structure constants of the  $N = 2$  super- $W_\infty(\lambda)$  algebra. Using these structure constants one can give a closed expression for the quantum corrected transformation rules of the gauge fields  $A_{(s)}$ . The explicit form of these transformations is given in Eq. (84).

## 5. The $N = 2$ Super- $W_\infty(\lambda)$ Algebra

In this section we describe the structure of the  $N = 2$  super- $W_\infty(\lambda)$  algebra. This section follows the analysis of Ref. 53 with the notation adapted to this paper. The quantum  $N = 2$  super- $W_\infty(\lambda)$  algebra can be described as the algebra formed by arbitrary positive powers of the superspace differential operator  $D$ . The explicit expressions for the differential operators are given by ( $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ )

$$L_\lambda^{(s)}(k_{(s)}) = \sum_{i=0}^{2s-1} A^i(s, \lambda) (D^{2s-i-1} k_{(s)}) D^i, \quad (46)$$

where  $\lambda$  is the conformal weight. The summation index  $i$  takes only integer values. The parameter  $k_{(s)}$  is commuting (anticommuting) for half-integer (integer)  $s$ . The algebra formed by the  $L^{(s)}$  operators generalize the super-Virasoro algebra which is generated by

$$L_\lambda^{(3/2)}(k_{(3/2)}) = -k_{(3/2)} D^2 - \frac{1}{2} (D k_{(3/2)}) D - \lambda (D^2 k_{(3/2)}). \quad (47)$$

The coefficients  $A^i(s, \lambda)$  are given by<sup>53</sup>

$$A^i(s, \lambda) = (-)^{[s+1/2]+(2s+1)(i+1)+1} \frac{[s-1/2]!}{[i/2]![s-i/2-1/2]!} \times \frac{([i/2+1/2]+2\lambda)_{[s]-[i/2+1/2]}}{([s+i/2+1/2])_{2s-[s+i/2+1/2]}}, \quad (48)$$

where  $(a)_n \equiv (a+n-1)!/(a-1)!$  and  $[a]$  denotes the integer part of  $a$ . These coefficients are fixed by the requirement that the  $L^{(s)}$  form nondecomposable representations of the  $osp(1, 2)$  subalgebra of the super-Virasoro algebra. This  $osp(1, 2)$  subalgebra is generated by the differential operators  $L_\lambda^{(3/2)}$  with the parameters  $k_{(3/2)}$  satisfying

$$D^5 k_{(3/2)} = 0. \quad (49)$$

It is possible to define a field-theoretic representation of the super- $W_\infty(\lambda)$  algebra in terms of a superconformal  $BC$  system. In this representation we have two conformal superfields, a commuting field  $B$  and an anticommuting field  $C$ , with conformal weights  $\lambda$  and  $\frac{1}{2} - \lambda$ , respectively. When subjected to their field equations, these fields decompose according to  $B(z, \theta) = \beta(z) + \theta b(z)$  and  $C(z, \theta) = c(z) + \theta \gamma(z)$ . Since  $\theta$  has weight  $-\frac{1}{2}$ , we find that  $b, c, \beta$  and  $\gamma$  have conformal weights  $\lambda + \frac{1}{2}, -\lambda + \frac{1}{2}, \lambda$  and  $-\lambda + 1$ . The supersymmetric action equals<sup>54</sup>

$$S = \frac{1}{\pi} \int d^2 z B \bar{D} C = \frac{1}{\pi} \int d^2 z \{ \beta \bar{\partial} \gamma + b \bar{\partial} c \}. \quad (50)$$

The generators of the super- $W_\infty(\lambda)$  algebra are related to the following conserved supercurrents of (quasi-)conformal spin  $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ :

$$W_\lambda^{(s)} = \sum_{i=0}^{2s-1} (-)^{[s-i/2]+[2s+1]_2[2s-i-1]_2} A^i(s, \lambda) D^{2s-i-1} \{ (D^i B) C \} \\ = \sum_{i=0}^{2s-1} \tilde{A}^i(s, \lambda) (D^i B) (D^{2s-i-1} C) \quad (51)$$

with the coefficients  $\tilde{A}^i(s, \lambda)$  given by<sup>53</sup>

$$\tilde{A}^i(s, \lambda) = \frac{1 + |2s|_2 |i+1|_2}{1 + |2s|_2} \frac{(-)^{[s]+[\frac{i}{2}]+[2s+1]_2[i+1]_2}}{(-[-s])_{[s]-|2s|_2[i]_2}} \begin{pmatrix} [s] - 1 + |2s|_2 |i+1|_2 \\ \left[ \frac{i}{2} \right] \end{pmatrix} \\ \times (2\lambda - [s])_{[\frac{i}{2}]+[2s+1]_2[i]_2} (-2\lambda - [s] + 1)_{[s]-[\frac{i}{2}]-[i]_2}. \quad (52)$$

In Eq. (51) a normal ordering with respect to the modes of  $B, C$  is understood. The supercurrent  $W_\lambda^{(s)}$  is commuting (anticommuting) for integer (half-integer)  $s$ . Each supercurrent contains a spin  $s$  and a spin  $s + \frac{1}{2}$  component current. The



normalization of the supercurrents is taken such that the current  $W_\lambda^{(s)}$  exactly generates the variation corresponding to the differential operator<sup>k</sup>  $L_\lambda^{(s)}$ :

$$\delta(k_{(s)})B(2) = L_\lambda^{(s)}(k_{(s)})B(2) = \frac{1}{2\pi i} \oint dZ_1 k_{(s)}(1) W_\lambda^{(s)}(1) B(2) \quad (53)$$

with the two-point function of  $B$  and  $C$  given in Eq. (57). The coefficients  $\tilde{A}^i(s, \lambda)$  can also be determined by the requirement that the supercurrents  $W_\lambda^{(s)}$  form  $N = 1$  superfields with respect to the  $osp(1, 2)$  subalgebra of the super-Virasoro algebra which is generated by  $W_\lambda^{(3/2)}$ . We note that the  $\tilde{A}^i(s, \lambda)$  satisfy the identity

$$\tilde{A}^i(s, \lambda) = (-)^{-[s]-1+|2s|_2|i|_2} \tilde{A}^{2s-i-1}\left(s, \frac{1}{2} - \lambda\right). \quad (54)$$

From this identity we see that the value  $\lambda = 1/4$  is special. We find that for that value of  $\lambda$  the coefficients  $\tilde{A}^i(s, 1/4)$  can be written as

$$\tilde{A}^i\left(s, \frac{1}{4}\right) = \frac{(-)^{|2s|_2+[\frac{1}{2}+\frac{1}{2}]+1}}{(1+|2s|_2)2^{2[s]-1}} \binom{2s-1}{i}. \quad (55)$$

It turns out that the currents  $\{W_\lambda^{(1)}, W_\lambda^{(3/2)}\}$  form an  $N = 2$  super-Virasoro algebra.<sup>54</sup> All currents fit into  $N = 2$  supermultiplets with respect to the  $osp(2, 2)$  subalgebra of this  $N = 2$  super-Virasoro algebra. This  $osp(2, 2)$  subalgebra is generated by the differential operators  $\{L_\lambda^{(1)}, L_\lambda^{(3/2)}\}$  with the parameters  $k_{(1)}, k_{(3/2)}$  satisfying

$$D^3 k_{(1)} = D^5 k_{(3/2)} = 0. \quad (56)$$

This generalizes Eq. (7) to the quantum algebra. The resulting  $N = 2$  combinations are  $\{W_\lambda^{(s)}, W_\lambda^{(s+1/2)}\}$  for integer  $s$  and  $W_\lambda^{(1/2)}$ , where  $W_\lambda^{(1/2)}$  constitutes a so-called  $N = 2$  scalar multiplet.

The two-point function of the superfields  $B(Z)$  and  $C(Z)$  is equal to<sup>54</sup>

$$\langle C(Z_1)B(Z_2) \rangle = \frac{\theta_{12}}{z_{12}}. \quad (57)$$

The  $N = 1$  super-Virasoro algebra generated by  $W_\lambda^{(3/2)}$  is defined by the following operator product expansion

$$\begin{aligned} W_\lambda^{(3/2)}(1)W_\lambda^{(3/2)}(2) \sim & \frac{3}{2} \frac{\theta_{12}W_\lambda^{(3/2)}}{z_{12}^2} - \frac{1}{2} \frac{D_2W_\lambda^{(3/2)}}{z_{12}} + \frac{\theta_{12}\partial_2W_\lambda^{(3/2)}}{z_{12}} \\ & + 2 \frac{\lambda - \frac{1}{4}}{z_{12}^3} + \text{regular}. \end{aligned} \quad (58)$$

Here we use the super-Taylor expansion

<sup>k</sup>For simplicity we will set  $\hbar = 1$  everywhere in this section.

$$W(1) = W(2) + \theta_{12} D_2 W + z_{12} \partial_2 W + \theta_{12} z_{12} \partial_2 D_2 W + \frac{1}{2} z_{12}^2 \partial_2^2 W + \dots \quad (59)$$

The OPE expansion of two general supercurrents  $W_\lambda^{(s)}$  and  $W_\lambda^{(t)}$  is given by the expression<sup>l</sup>

$$W_\lambda^{(s)}(1) W_\lambda^{(t)}(2) \sim \sum_{u=\frac{1}{2}}^{s+t-\frac{1}{2}} f_{st}^u(D_1, D_2; \lambda) \frac{\theta_{12} W_\lambda^{(s+t-u)}(2)}{z_{12}} + c(s, t; \lambda) \frac{\theta_{12}^{2(s+t)|_2}}{(z_{12})^{s+t+\frac{1}{2}|2(s+t)|_2}}. \quad (60)$$

The structure functions  $f_{st}^u(D_1, D_2; \lambda)$  are polynomials in the supercovariant derivatives of degree  $2u - 1$ :

$$f_{st}^u(D_1, D_2; \lambda) = f_{st}^u(\lambda) \sum_{i=0}^{2u-1} M_{st}^u(i) D_1^i D_2^{2u-i-1}. \quad (61)$$

The functions  $M_{st}^u(i)$  are fixed by the requirement of  $osp(1, 2)$  covariance. They are given by<sup>m</sup>

$$M_{st}^u(i) = (-)^{i(2s+2u+1)+[u-i/2]+|2s+1|_2|i|_2} \frac{[u-1/2]!}{[i/2]![u-i/2-1/2]!} \times ([2s-u+1/2])_{[u-i/2-1/2]+|2u+1|_2|2u-i-1|_2} \times ([2t-u+1/2])_{[i/2]+|2u+1|_2|i|_2}. \quad (62)$$

The structure constants  $f_{st}^u(\lambda)$  can be explicitly computed and are given by the expression<sup>53</sup>

$$f_{st}^u(\lambda) = F_{st}^u(\lambda) + (-)^{[-u-1/2]+4(s+u+1)(t+u+1)} F_{st}^u(1/2 - \lambda) \quad (63)$$

with

$$F_{st}^u(\lambda) = (-)^{2s+|2s|_2|2t+1|_2|2t+1|_2|2u-1|_2+|2s+2t+2u+1|_2} \times (-)^{[s+t-u-1/2]} \frac{(2s+2t-2u-1)!}{(2s+2t-[u+1/2]-1)!} \times \sum_{i=0}^{2s-1} \sum_{j=0}^{2t-1} \delta(i+j-2s-2t+2u+1) \times A^i\left(s, \frac{1}{2} - \lambda\right) A^j(t, \lambda) (-)^{2i(s+t-u+1/2)}. \quad (64)$$

<sup>l</sup>For a number of representative cases the explicit form of the OPE expansions is given in App. A.

<sup>m</sup>Strictly speaking the statement is that the functions

$$\tilde{M}_{st}^u(i) = (-)^{[i/2+1/2]+[u-i/2]+|2s+1|_2|i|_2} M_{st}^u(i)$$

are fixed by the  $osp(1, 2)$  covariance. The extra  $i$ -dependent sign factor arises from the particular way we have rewritten the commutator-algebra calculation of Ref. 53 in terms of the above OPE expansions.

Finally, the central charge  $c(s, t; \lambda)$  is given by<sup>53</sup>

$$\begin{aligned}
 c(s, t; \lambda) = & \sum_{i=0}^{2s-1} \sum_{j=0}^{2t-1} (-)^{|i|_2|2s-i+j|_2+|j|_2+[j/2+1/2]+[t-j/2]} \\
 & \times \{ |2(s+t)|_2 + |2(s+t)+1|_2 (|2s+1|_2|i+j+1|_2 + |2s|_2|i+j|_2) \} \\
 & \times ([j/2] + [s-i/2-1/2] + |2(s+t)|_2|2s-i+1|_2|j|_2)! \\
 & \times ([i/2] + [t-j/2-1/2] + |2(s+t)|_2|2t-j+1|_2|i|_2)! \\
 & \times \tilde{A}^i(s, \lambda) \tilde{A}^j(t, \lambda).
 \end{aligned} \tag{65}$$

In particular we find  $c(3/2, 3/2; \lambda) \equiv -c/6 = 2(\lambda - 1/4)$ , or

$$c = -12(\lambda - 1/4), \tag{66}$$

where  $c$  is the usual central charge parameter of the Virasoro algebra. This Virasoro algebra is defined by the expansion  $W_\lambda^{(3/2)} = 1/2iG + \theta T$  such that  $T$  satisfies the standard Virasoro algebra

$$T(1)T(2) \sim \frac{2T}{(z_1 - z_2)^2} + \frac{\partial_2 T}{z_1 - z_2} + \frac{c/2}{(z_1 - z_2)^4}. \tag{67}$$

Note that for  $\lambda = 0$  we have  $c = 3$  as one would expect for a supersymmetric  $BC$  system. Other choices of  $\lambda$  give other choices of the central charge but they refer to other Virasoro subalgebras. Note that the  $BC$  system is a particular  $c = 3$  representation<sup>n</sup> of a more general class of algebras with an arbitrary central charge parameter  $c$ ,<sup>10</sup> which occurs linearly in all central terms. Therefore, in the  $\lambda = 0$  basis the central charges  $c(s, t)$  of the higher spins are related to  $c$  as

$$c(s, t) = -\frac{1}{6} \frac{c(s, t; 0)}{c(3/2, 3/2; 0)} c. \tag{68}$$

From the above expressions one can derive some general properties of the structure constants and the central charges. First of all, we find

$$f_{3/2s}^{1/2}(\lambda) = f_{3/2s}^1(\lambda) = f_{3/2s}^2(\lambda) = f_{3/2s}^{3/2}(\lambda) = 0 \tag{69}$$

in agreement with the  $osp(1, 2)$  covariance of the equations. The maximum value  $s_{\max}$  of the spin arising at the right-hand side of the OPE between two currents of spin  $s$  and  $t$  ( $s, t \neq 1/2$ ) is given by

$$\begin{aligned}
 s_{\max} &= s + t - \frac{1}{2} & (s, t \text{ integer}), \\
 s_{\max} &= s + t - \frac{3}{2} & (\text{all other cases}).
 \end{aligned} \tag{70}$$

<sup>n</sup>Actually, as will be explained later in this section, the  $BC$  system provides us with a  $c = 3$  as well as with a  $c = -3$  representation of the general algebra.

These maximum values of the spin can be understood from the fact that within the so-called wedge subalgebra one can use the addition rules for the spins according to the  $osp(1, 2)$  algebra. This wedge subalgebra can be defined by the following restrictions on the parameters:<sup>53</sup>

$$D^{4s-1}k_{(s)} = 0. \quad (71)$$

Finally, from the general formulae given above one can deduce that the structure constants and the central charges satisfy the identities:

$$\begin{aligned} f_{st}^u(\lambda) &= (-)^{[-u-\frac{1}{2}]+4(s+u+1)(t+u+1)} f_{st}^u\left(\frac{1}{2}-\lambda\right), \\ c(s, t; \lambda) &= (-)^{[s+t]+[2s]_2[2t]_2} c(t, s; \lambda) = -c\left(t, s; \frac{1}{2}-\lambda\right). \end{aligned} \quad (72)$$

On the basis of these relations one can show that the super- $W_\infty(\lambda)$  and super- $W_\infty(1/2-\lambda)$  algebras (*without* the central terms) are equivalent to each other. To be precise, the form of the OPE expansions (60) does not change if one replaces  $\lambda$  everywhere by  $1/2-\lambda$  and furthermore redefines the currents with a factor  $(-)^{[s]+1}$ , i.e.

$$W_\lambda^{(s)} \rightarrow (-)^{[s]+1} W_{1/2-\lambda}^{(s)}. \quad (73)$$

This equivalence is only true at the level of single contractions or, equivalently, Poisson brackets. It ceases to be true if one includes the central terms in (60), which correspond to double contractions. In fact, one finds that under the map (73) all the central terms change sign. This change of sign can be understood as follows. Under the map  $\lambda \rightarrow 1/2-\lambda$  one effectively interchanges the role of the bosonic  $\beta\gamma$  system and the fermionic  $bc$  system in the action. Both before as well as after this interchange one can realize the same super- $W_\infty$  algebra (i.e. with identical structure constants). However, since bosons have been interchanged with fermions, the contribution to the central charge changes sign.

As an example, consider the  $BC$  system at  $\lambda = 0$ . On this system one can realize a  $c = 3$  representation of the  $N = 2$  super- $W_\infty(0)$  algebra. The statement now is that on a  $BC$  system with  $\lambda = 1/2$  one can realize a  $c = -3$  representation of exactly the same  $N = 2$  super- $W_\infty(0)$  algebra.

Based on the relations given above one can discuss different truncations of super- $W_\infty(\lambda)$ . We will briefly discuss here two truncations that are possible at  $\lambda = 0$  (or, equivalently,  $\lambda = 1/2$ ) and  $\lambda = 1/4$ , respectively.

For  $\lambda = 0$  one can reduce the super- $W_\infty(\lambda)$  algebra to an algebra with  $N = 1$  supersymmetry. For that value of  $\lambda$  one can do two special things. First of all, it turns out that for  $\lambda = 0$  the  $s = 1/2$  generator can be truncated away from the algebra. The reason for this is that for  $\lambda = 0$  the  $B$  superfield only occurs as  $DB$  in the expressions for the supercurrents, except in the  $s = 1/2$  supercurrent. Therefore, in the OPE of two currents  $W_0^{(s)}$  and  $W_0^{(t)}$  with  $s, t \neq 1/2$  the superfield

$B$  will only occur as  $DB$ . For general values of  $\lambda$  this property is reflected in the fact that in the right-hand side of the OPE the  $s = 1/2$  supercurrent  $W_\lambda^{(1/2)}$  always is multiplied by the second-order Casimir  $C_2$  of  $osp(1, 2)$  which is  $C_2 = \lambda(\lambda - 1/2)$ . Consequently, for  $\lambda = 0$  the  $s = 1/2$  supercurrent can be consistently truncated away from the algebra, giving the  $N = 2$  super- $W_\infty$  algebra of Ref. 10.

Secondly, for  $\lambda = 0$  the superfields  $C$  and  $DB$  both have conformal weight  $s = 1/2$ . One can therefore perform a further truncation of the algebra by identifying  $C$  with  $DB$ :

$$C \equiv DB. \quad (74)$$

Note that this identification can only be implemented after having discarded the  $s = 1/2$  generator since only then the  $B$  superfield will always occur as  $DB$ . The effect of the above truncation is that all supercurrents  $W_0^{(s)}$  with  $s$  or  $s + 1/2$  odd vanish identically. One is then left with the supercurrents  $W_0^{(s)}$  with  $s$  or  $s + 1/2$  even only. They generate an algebra with  $N = 1$  supersymmetry which we will denote with  $N = 1$  super- $W_\infty$ . The expressions for the currents are given by

$$W_0^{(s)} = \sum_{i=1}^{[s]} \tilde{A}^i(s, 0)(D^i B)(D^{2s-i} B) - \frac{1}{2}|2s + 1|_2 \tilde{A}^s(s, 0)(\partial^{s/2} B)^2. \quad (75)$$

We have chosen here the normalization of the currents such that the nonzero structure constants are exactly the same as the ones of the  $N = 2$  super- $W_\infty(0)$  algebra. We use here the two-point function for the scalar superfield  $B$ :

$$\langle DB(Z_1)B(Z_2) \rangle = \frac{\theta_{12}}{z_{12}}. \quad (76)$$

The expressions  $\tilde{c}(s, t)$  for the central charges, however, are twice as small, i.e.

$$\tilde{c}(s, t) = \frac{1}{2}c(s, t; 0). \quad (77)$$

In particular, we find that  $\tilde{c}(3/2, 3/2) = 3/2$  as one would expect for a single real scalar superfield  $B$ .

It is interesting to note that the  $\lambda = 0$  truncation, described above, is the beginning of a whole series of truncations that take place for  $\lambda = 0, -1/2, -1, -3/2, \dots$ .<sup>o</sup> For instance, for  $\lambda = -1/2$  one can first truncate away the  $s = 1/2$  generator as well as the  $\{W_{-1/2}^{(1)}, W_{-1/2}^{(3/2)}\}$  multiplet. This leads to an  $N = 2$  algebra that starts with the  $\{W_{-1/2}^{(2)}, W_{-1/2}^{(5/2)}\}$  multiplet. One can then truncate the  $N = 2$  supersymmetry to an  $N = 1$  supersymmetry by making the identification

$$C = \partial DB. \quad (78)$$

Note that for  $\lambda = -1/2$  indeed the conformal weights of  $C$  and  $\partial DB$  coincide. The general pattern is then as follows. For  $\lambda = -M/2$  ( $M = 0, 1, 2, \dots$ ) one can truncate

<sup>o</sup>A similar set of truncations in the bosonic case has been discussed in the second reference of Ref. 22, and from a different point of view in Ref. 55.

away the  $N = 2$  multiplets  $W_{-M/2}^{(1/2)}, \{W_{-M/2}^{(1)}, W_{-M/2}^{(3/2)}\}, \dots, \{W_{-M/2}^{(M)}, W_{-M/2}^{(M+1/2)}\}$ . The truncation to  $N = 1$  supersymmetry is then achieved by making the identification

$$C = \partial^M DB. \tag{79}$$

Note that such identifications lead to higher-derivative actions for the  $B$  superfield:

$$S = \frac{1}{\pi} \int d^2 Z (\bar{D}B) \partial^M DB. \tag{80}$$

Since all these truncations (except for  $M = 0$ ) lead to algebras that do not contain a Virasoro subalgebra we will not consider them further in this paper.

In contrast to the  $\lambda = 0$  truncation, the  $\lambda = 1/4$  truncation preserves the  $N = 2$  supersymmetry of the super- $W_\infty(\lambda)$  algebra. On the basis of the symmetry properties of the structure constants given in Eq. (72) one deduces that the structure constants  $f_{st}^u(1/4)$  vanish identically, whenever  $[-u - 1/2] + 4(s + u + 1)(t + u + 1)$  is odd. This enables one to show that for  $\lambda = 1/4$  one can perform a consistent truncation of the super- $W_\infty(1/4)$  algebra such that one retains the  $\{W_{1/4}^{(s)}, W_{1/4}^{(s+1/2)}\}$   $N = 2$  supermultiplets with  $s$  odd only. This truncated algebra is related to the symplectic higher-spin superalgebra of Refs. 56, 57. We note that the classical version of the truncated  $N = 2$  super- $W_\infty(1/4)$  algebra is the  $N = 2$  super- $w_{\infty/2}$  algebra which we introduced in Sec. 3.

We have shown that the quantum  $N = 2$  super- $W_\infty(\lambda)$  algebra can be truncated for  $\lambda = 0$  and  $\lambda = 1/4$ . In Sec. 3 we have discussed similar truncations of the classical  $N = 2$  super- $w_\infty$  algebra. We should stress that every truncation of the quantum algebra corresponds to a truncation of the corresponding classical algebra but that the reverse is not true: the classical algebra allows truncations that have no quantum analogue. In the table below we have given the classical limits of the  $\lambda = 0$  and  $\lambda = 1/4$  truncated quantum algebras discussed above.

Table 2. Truncations of some classical  $w_\infty$  superalgebras and their quantum extensions.

Classical algebra	Quantum algebra
$N = 2$ super- $w_{1/2+\infty}$	$N = 2$ super- $W_\infty(\lambda)$
$N = 2$ super- $w_\infty$	$N = 2$ super- $W_\infty$
$N = 2$ super- $w_{\infty/2}$	$N = 2$ super- $W_\infty(1/4)$
$N = 1$ super- $w_\infty$	$N = 1$ super- $W_\infty$

With the OPE expansions given in the appendix, one may verify the consistency of the truncations.

We should note that in the generic case the currents  $W_\lambda^{(s)}$  are quasi-primary but not primary with respect to the  $N = 1$  super-Virasoro algebra generated by  $W_\lambda^{(3/2)}$ . Only the currents  $W_\lambda^{(1)}, W_{1/4}^{3/2}$  and  $W_0^{(2)}$  (or  $W_{1/2}^{(2)}$ ) are  $N = 1$  primary. The reason

for this is that we preferred to work with a realization of the currents in terms of bilinears of the  $B, C$  superfields. In this realization the  $N = 2$  super- $W_\infty(\lambda)$  algebra is a linear algebra. On the other hand, in most of the literature on nonlinear  $W$  algebras a basis is used where all generators are primary. In our case, we also could have used a primary basis but, to represent the currents, we should allow not only bilinears in  $B, C$  but also terms which are quadrilinear and of higher order in  $B, C$ . In the primary basis the super- $W_\infty(\lambda)$  algebra is nonlinear.

To illustrate the above point, we consider the first two currents beyond the  $N = 2$  super-Virasoro algebra, i.e.  $W_\lambda^{(2)}$  and  $W_\lambda^{(5/2)}$ . These currents are given in terms of bilinears of  $B, C$  (see App. A). The current  $W_\lambda^{(2)}$  is only  $N = 1$  primary for  $\lambda = 0, 1/2$ , whereas  $W_\lambda^{(5/2)}$  is not primary for any value of  $\lambda$ . We will now show how, by allowing also terms of higher-order in  $B$  and  $C$ , one can construct currents  $W_\lambda^{(2)'}$  and  $W_\lambda^{(5/2)'}$ , which are  $N = 1$  primary for any value of  $\lambda$ .<sup>P</sup> Starting from the most general polynomial in  $B$  and  $C$  we find the following expressions for  $W_\lambda^{(2)'}$  and  $W_\lambda^{(5/2)'}$ :

$$\begin{aligned}
 W_\lambda^{(2)'} = & +\alpha(2\lambda - 1)(\partial DB)C \\
 & +\alpha(2\lambda + 1)(\partial B)DC \\
 & +(\alpha + 2\beta(3\lambda - 2))(DB)\partial C \\
 & +\beta(6\lambda - 1)B\partial DC \\
 & +2(\alpha\lambda - \beta(3\lambda - 2))B(DB)CDC \\
 & +(\alpha\lambda(4\lambda + 1) - \beta(12\lambda^2 - 11\lambda + 1))/2(2\lambda - 1)B^2(DC)DC \\
 & +(\alpha\lambda + \beta(3\lambda - 1))B^2C\partial C, \\
 W_\lambda^{(5/2)'} = & +(-1 + 2\lambda)(-1 + 3\lambda)(-3 + 4\lambda)(\partial^2 B)C \\
 & +2(-1 + 3\lambda)(-3 + 4\lambda)(\partial DB)DC \\
 & -2(-5 + 34\lambda - 56\lambda^2 + 24\lambda^3)(\partial B)\partial C \\
 & +(-1 + 6\lambda)(DB)\partial DC \\
 & -\lambda(-1 + 6\lambda)(-7 + 12\lambda)B\partial^2 C \\
 & +2(-1 + 2\lambda)(2 - 23\lambda + 24\lambda^2)B(\partial B)CDC \\
 & -2(-1 + 2\lambda)(-4 + 3\lambda)(-1 + 4\lambda)B(DB)C\partial C \\
 & +2(-1 - 2\lambda + 6\lambda^2)B(DB)(DC)DC \\
 & +4\lambda(-1 - 2\lambda + 6\lambda^2)B^2(DC)\partial C \\
 & +6\lambda(-1 + 2\lambda)(-1 + 6\lambda)B^2C\partial DC,
 \end{aligned} \tag{81}$$

<sup>P</sup>A similar discussion in the case of a bosonized  $BC$  system has been given in Ref. 58.

where  $\alpha$  and  $\beta$  are two arbitrary parameters. Note that the  $B^2(DC)DC$  term in the expression for  $W_\lambda^{(2)'}$  is singular for  $\lambda = 1/2$ . For that value of  $\lambda$  the expression for  $W^{(2)'}$  is given by

$$\begin{aligned} W_{1/2}^{(2)'} &= +\alpha'((\partial B)DC + (DB)\partial C - B\partial DC) \\ &+ \beta' B^2(DC)DC \end{aligned} \quad (82)$$

with  $\alpha'$  and  $\beta'$  arbitrary.

The above expressions for  $W_\lambda^{(2)'}$  and  $W_\lambda^{(5/2)'}$  are not necessarily primary with respect to the  $N = 2$  super-*Virasoro* algebra generated by  $\{W_\lambda^{(1)}, W_\lambda^{(3/2)}\}$ . It turns out that this is only the case for  $\lambda = 1/2$  and  $\alpha' = 2\beta' = -2$ . In particular, it is not true for  $\lambda = 0$ . We expect that the following picture extends to the higher-spin generators but we have not proven this. One can define  $N = 1$  primary currents  $W_\lambda^{(s)'}$  for arbitrary values of  $\lambda$ . Only for particular values of  $\lambda$  can one define  $N = 2$  primary superfields  $\{W_\lambda^{(s)}, W_\lambda^{(s+1/2)}\}$  ( $s \geq 2$  integer). We note that the  $N = 2$  superfield  $W_\lambda^{(1/2)}$  plays a special role. From the OPE expansions given in the appendix it is clear that, without redefining the  $N = 2$  super-*Virasoro* generators, one cannot define an  $N = 1$  or  $N = 2$  primary current with spin  $s = 1/2$ .<sup>9</sup>

The quasi-primary currents  $\{W_\lambda^{(2)}, W_\lambda^{(5/2)}\}$  and the  $N = 1$  primary currents  $\{W_\lambda^{(2)'}, W_\lambda^{(5/2)'}\}$  are related to each other by means of a nonlinear redefinition of the generators of the  $N = 2$  super- $W_\infty(\lambda)$  algebra. In general, this redefinition involves the  $s = 1/2$  generator. One can show however that for  $\lambda = 1/2$  and  $\alpha' = 2\beta' = -2$  the  $s = 1/2$  generator is absent. The nonlinear redefinitions for this case are given in Eq. (115).

In order to make contact with the results of Sec. 4 where we quantized the  $N = 2$   $w_\infty$ -supergravity theory one should replace the  $B, C$  superfields by two scalar superfields  $\phi, \bar{\phi}$  by applying the superbosonization rules<sup>59</sup>

$$B = e^\phi, \quad C = e^{-\phi} D\bar{\phi}. \quad (83)$$

Using these superbosonization rules one can show that the quantum currents described in this section in terms of higher-derivative bilinears in  $B$  and  $C$  are equivalent to nonlinear expressions in terms of  $D\phi, D\bar{\phi}$  and supercovariant derivatives thereof. For instance, one may verify that for  $\lambda = 0$  one exactly finds, starting from the  $BC$  currents given in Eq. (51) the quantum currents given in Eq. (36). For more details we refer to Ref. 50.

The structure constants of the  $N = 2$  super- $W_\infty(\lambda)$  algebra can also be used to give a closed expression for the quantum-corrected transformation rules of the gauge fields  $A_{(s)}$ . These transformation rules follow from Eqs. (39) and (40). In applying Eq. (40) we now use the structure constants of the full quantum  $N = 2$  super- $W_\infty(\lambda)$  algebra. The final result is given by

<sup>9</sup>We thank J. de Boer for a discussion on this point.



$$\delta A_{(s)} = \bar{D}k_{(s)} - \sum_{u,t=\frac{1}{2}}^{s+u-\frac{1}{2}} A_{(t)} k_{(s-t+u)} \tilde{f}_{s-t+u,t}^u(\bar{D}_k, \bar{D}_A; \lambda). \quad (84)$$

Here it is understood that  $k_{(s)} = 0$  for  $s \leq 0$ . Furthermore the derivatives  $\bar{D}_k$  ( $\bar{D}_A$ ) act on  $k_{(s-t+u)}(A_{(t)})$  only. The  $\tilde{f}$  structure constants are given by

$$\begin{aligned} \tilde{f}_{st}^u(\bar{D}_1, \bar{D}_2; \lambda) &= f_{st}^u(\lambda) \sum_{i=0}^{2u-1} (-)^{[i/2+1/2]+[u-i/2]+|2u-1|_2+|2s+1|_2} \\ &\times M_{st}^u(i) \overleftarrow{D}_1^i \overleftarrow{D}_2^{2u-i-1}. \end{aligned} \quad (85)$$

We note that the explicit  $i$ -dependent sign factors in this equation are due to the fact that we work with the functions  $M_{st}^u(i)$  instead of the  $\tilde{M}_{st}^u(i)$  (see also the footnote before Eq. (62).

## 6. Universal Anomalies

Having canceled the matter-dependent anomalies in Sec. 4, we discuss in this section the universal anomalies. These anomalies arise from diagrams with only external gauge fields. We will see that the universal anomalies are related to the central terms in the quantum algebra. Since the central terms are numbers, not containing any quantum currents, the cancellation of the universal anomalies requires a mechanism different from that of the matter-dependent anomalies. We have already seen that the matter-dependent anomalies can be canceled by an appropriate renormalization of the gauge field transformation rules. In this section we will see that the cancellation of the universal anomalies requires finding a  $c = 0$  representation of the quantum algebra in terms of matter and ghosts fields.

Following Refs. 60, 36 we first derive an anomalous Ward identity for the universal anomalies. Considering only diagrams with external gauge fields, the effective action is, in terms of operator expectation values,

$$e^{-\Gamma(A_{(s)})} = \left\langle \exp \left( -\frac{1}{\pi} \int \sum_s A_{(s)} W^{(s)} \right) \right\rangle. \quad (86)$$

Varying this equation with respect to  $A_{(s)}(Z_1)$  and differentiating with respect to  $\bar{Z}_1$ , one finds

$$\bar{D}_1 \frac{\delta \Gamma}{\delta A_{(s)}(Z_1)} = \frac{1}{\pi} \left\langle \bar{D}_1 W^{(s)}(Z_1) \exp \left( -\frac{1}{\pi} \int \sum_t A_{(t)} W^{(t)} \right) \right\rangle e^\Gamma. \quad (87)$$

Using the OPE expansion of the super- $W_\infty(\lambda)$  algebra, we may calculate

$$\begin{aligned}
 & \bar{D}_1 W^{(s)}(Z_1) \exp \left( -\frac{1}{\pi} \int \sum_t A_{(t)} W^{(t)} \right) \\
 &= -\frac{1}{\pi} \int d^2 Z_2 A_{(t)}(Z_2) \bar{D}_1 \left( \sum_{u=\frac{1}{2}}^{s+t-\frac{1}{2}} f_{st}^u(D_1, D_2; \lambda) \frac{\theta_{12} W^{(s+t-u)}(2)}{z_{12}} \right. \\
 & \quad \left. + c(s, t; \lambda) \frac{\theta_{12}^{[2(s+t)]_2}}{(z_{12})^{s+t+\frac{1}{2}[2(s+t)]_2}} \right) \\
 & \quad \times (-)^{|2t|_2|2s+1|_2} \exp \left( -\frac{1}{\pi} \int \sum_t A_{(t)} W^{(t)} \right). \tag{88}
 \end{aligned}$$

Since  $\bar{D}_1 \frac{\theta_{12}}{z_{12}} = \pi \Delta(Z_1 - Z_2)$ , we may perform the  $Z_2$  integration. If we now multiply the whole equation with the factor

$$\sum_{s \geq 1/2} \int d^2 Z_1 (-)^{|2s|_2} k_{(s)}(1) \tag{89}$$

we find the following anomalous Ward identity for  $N = 2$   $W_\infty$  supergravity:

$$\delta_k \Gamma = \sum_{s, t \geq \frac{1}{2}} \frac{\hat{c}(s, t; \lambda)}{\pi} \left\langle \int d^2 Z k_{(s)} D^{2(s+t)-1} A_{(t)} \right\rangle, \tag{90}$$

where  $\hat{c}(s, t; \lambda)$  is related to the central charge  $c(s, t; \lambda)$  as

$$\hat{c}(s, t; \lambda) = \frac{(-)^{|2s+1|_2|2t|_2}}{(s+t+\frac{1}{2}[2(s+t)]_2-1)!} c(s, t; \lambda). \tag{91}$$

In deriving this equation we have used that the transformation rule of  $A_{(s)}$  is given by (84). Thus we see that the effective action is not invariant under spin- $s$  super- $W_\infty$  transformations, on account of the anomalous terms on the right-hand side. We note that these terms are exactly the ones that arise from calculating the central charges in the quantum algebra. Thus, every central term in the quantum algebra corresponds via the above expression to a universal anomaly.

One might hope that the universal anomalies can be canceled by integrating over all (component) higher-spin gauge fields. In general, this integration gives rise to ghosts which contribute to the central charge in the Virasoro sector as

$$c_{\text{gh}} = \lim_{N \rightarrow \infty} \sum_{s=\frac{1}{2}}^N c_{\text{gh}}(s) \tag{92}$$

with

$$c_{\text{gh}}(s) = 2(-)^{2s+1}(6s^2 - 6s + 1). \tag{93}$$

As discussed in Refs. 61, 62, one may define the above (divergent) sum by using an appropriate zeta function regularization. Using this regularization one can calculate  $c_{\text{gh}}$  for the different  $W_\infty$  algebras. In particular, in Ref. 62 it was found that for the  $N = 2$  super- $W_\infty$  algebra (without the  $s = 1/2$  generator<sup>r</sup>) in the  $\lambda = 0$  basis the ghost contribution to the central charge in the Virasoro sector is given by

$$c_{\text{gh}} = 3. \quad (94)$$

The idea is now to cancel this ghost contribution by an equal (but with an opposite sign) contribution  $c_{\text{matter}}$  of the matter fields such that

$$c_{\text{total}} = c_{\text{gh}} + c_{\text{matter}} = 0. \quad (95)$$

Following the discussion below Eq. (72) we see that indeed we can achieve this by taking a  $BC$  system in the  $\lambda = 1/2$  basis<sup>s</sup> such that

$$c_{\text{matter}} = -3 \quad (96)$$

and hence  $c_{\text{total}} = 3 - 3 = 0$ . Note that using a  $BC$  system in the  $\lambda = 0$  basis, which has  $c_{\text{matter}} = +3$ , would not work. We therefore conclude that the remarkable anomaly cancellation which was found in the bosonic  $w_\infty$ -gravity theory<sup>36</sup> also takes place in the supersymmetric case.

## 7. Truncations

It is known that in the bosonic  $w_\infty$ -gravity theory there exists a so-called telescoping procedure which enables one to truncate the theory to a classical  $w_N$ -gravity theory containing only a finite number of higher-spin generators.<sup>30</sup> This procedure requires that some of the higher-spin currents can be expressed as products of lower-spin currents, and makes use of the specific representation of the  $w_\infty$  algebra. Consider a multiscalar realization in which the currents are given by<sup>30</sup>

$$w^{(s)} = \frac{1}{s} \text{tr}(\partial\phi)^s \quad s \text{ integer}, \quad (97)$$

where the trace is in the fundamental representation of  $U(N)$  or  $SU(N)$ . One may verify that these currents satisfy the bosonic  $w_\infty$  algebra. For  $s \geq N+1$  one runs out of independent Casimir invariants and therefore it is possible to write all currents  $w^{(s)}$  with  $s \geq N+1$  as products of the currents  $w^{(s)}$  with  $s \leq N$ . An extreme case is the one-scalar realization of  $w_\infty$ ,<sup>30</sup> where all currents can be expressed in terms of the  $s = 2$  current.

We have seen that in order to supersymmetrize  $w_\infty$ -gravity we are forced to work with a two-scalar realization of the bosonic  $w_\infty$  algebra. Together with their

<sup>r</sup>We conjecture that, if the  $s = 1/2$  generator is included, the ghost contribution to the central charge vanishes.

<sup>s</sup>Note that this is consistent with the fact that only for  $\lambda = 0, 1/2$  one can close the quantum algebra without the  $s = 1/2$  generator.

fermionic partners they form the components of the two scalar superfields  $\phi$  and  $\bar{\phi}$ . In this section we will discuss the telescoping procedure in this realization, and discuss the corresponding mechanism at the quantum level. The first example of a truncation beyond ordinary (spin-2) supergravity would give an  $N = 2$   $W_3$ -supergravity theory. We will be mainly dealing with this case as a specific example.

At first sight it looks as if there is no telescoping procedure for the realization we are working with. From the expressions (10) for the classical currents, we deduce that all currents are linear in  $\bar{\phi}$ . Hence there is no way to write a single current as the product of lower-spin currents. However, it turns out that nonlinear relations, which do not contain terms linear in the currents, are possible.

We will first discuss the situation using an arbitrary  $\lambda$  basis. After that we will restrict ourselves to the case where the  $s = 1/2$  current can be consistently truncated away. As we have seen in Sec. 5, this forces us to use the  $\lambda = 0$  or  $\lambda = 1/2$  basis. We will see that the two different choices of  $\lambda$  lead to inequivalent results. Our strategy is to first consider identities between the classical currents  $w^{(s)}$  and then to consider their quantum extension, giving identities between the quantum currents  $W^{(s)}$ .

In discussing identities between the classical currents  $w^{(s)}$  we should distinguish between those which hold independently of the specific representation one is using and those which are representation-dependent. As an example of representation-independent identities we give here the set

$$w^{(s)}w^{(t)} - (-)^{|2s|_2|2t|_2}w^{(t)}w^{(s)} = 0. \quad (98)$$

Note that these include the relations  $w^{(s)}w^{(s)} = 0$  for half-integer  $s$ . It turns out that in our specific representation the above set of identities can be replaced by the following stronger conditions:

$$\begin{aligned} w^{(s)}w^{(t)} &= 0 & s, t \geq 1, \\ w^{(1/2)}w^{(s)} &= 0 & -s \text{ integer}, \\ w^{(1/2)}w^{(s)} &= \frac{1}{2}w^{(1/2)}Dw^{(s-1/2)} & s \text{ half-integer}. \end{aligned} \quad (99)$$

There are more identities, which involve derivatives of the currents.

To explain the situation in a general  $\lambda$  basis we consider all representation-dependent identities up to a total spin  $s = 2$ . We find that the independent relations are given by

$$\begin{aligned} w^{(1/2)}w^{(1)} &= 0, \\ w^{(1/2)}w^{(3/2)} &= \frac{1}{2}w^{(1/2)}Dw^{(1)}, \\ w^{(1)}w^{(1)} &= 0. \end{aligned} \quad (100)$$

Of course at any spin one can find dependent identities, either by taking derivatives of lower-spin identities or by multiplying a lower-spin identity with a current. In addition to (100), we find the following dependent identity:

$$Dw^{(1/2)}w^{(1)} - w^{(1/2)}Dw^{(1)} = 0. \quad (101)$$

The above classical relations cannot be used to express a higher-spin current in terms of a product of lower-spin currents. Therefore the classical telescoping procedure as discussed in Ref. 30 does not exist in this realization.

The situation changes drastically in the quantum case. In fact, we will now show that a quantum telescoping procedure does exist. It is to be expected that, when quantizing the  $N = 2$   $w_\infty$ -supergravity theory, the above classical relations receive quantum corrections proportional to Planck's constant  $\hbar$ . Of course, one should also replace the classical currents by the quantum currents. For instance, the representation-independent identities given in (98) deform into the quantum identities<sup>3 t</sup>

$$(W^{(s)}W^{(t)}) - (-)^{|2s|_2|2t|_2}(W^{(t)}W^{(s)}) = \sum_{1,2,\dots} \frac{(-)^{r+1}}{r!} \partial^r \{W^{(s)}W^{(t)}\}_r, \quad (102)$$

where  $\{W^{(s)}W^{(t)}\}_r$  is defined by the following terms in the OPE expansion:

$$W^{(s)}(1)W^{(t)}(2) = \sum_{r=1,2,\dots} \frac{\{W^{(s)}W^{(t)}\}_r(2)}{z_{12}^r} + (W^{(s)}W^{(t)})(2) + \dots \\ + \theta_{12} - \text{dependent terms} \quad (103)$$

and

$$(W^{(s)}W^{(t)})(2) = \frac{1}{2\pi i} \oint dZ_1 \frac{\theta_{12}}{z_{12}} W^{(s)}(1)W^{(t)}(2) \quad (104)$$

denotes the product of the supercurrents  $W^{(s)}$  and  $W^{(t)}$ , normal ordered with respect to the modes of  $W^{(s)}$  and  $W^{(t)}$ . We note that the chain rule for derivatives also applies to the normal ordered product

$$D(W^{(s)}W^{(t)}) = (DW^{(s)}W^{(t)}) + (-)^{|2s|_2}(W^{(s)}DW^{(t)}). \quad (105)$$

All the quantum identities discussed so far are representation-independent, and therefore are valid for *arbitrary* value of the central charge  $c$ .

We will now discuss the quantum extension of the representation-dependent identities.<sup>u</sup> We find the following quantum extension of the independent classical identities (100):<sup>v</sup>

$$\sqrt{\hbar}W^{(3/2)} = (W^{(1/2)}W^{(1)}) + 2\lambda\sqrt{\hbar}(W^{(1/2)}DW^{(1/2)}) \\ + \frac{1}{2}\sqrt{\hbar}DW^{(1)} - 2\lambda\hbar\partial W^{(1/2)},$$

<sup>t</sup>We thank K. Schoutens for a discussion on this point.

<sup>u</sup>To prove these identities, one may either use a representation in terms of  $\phi, \bar{\phi}$  or  $B, C$ .

<sup>v</sup>For simplicity, we will omit the subindex  $\lambda$  of the quantum currents everywhere in this section.

$$\begin{aligned}
 \sqrt{\hbar}W^{(2)} &= \frac{1}{2}(W^{(1)}W^{(1)}) - \frac{1}{3}(1-4\lambda)\sqrt{\hbar}(W^{(1)}DW^{(1/2)}) \\
 &\quad + \frac{1}{6}(1-4\lambda)\hbar\partial W^{(1)} + \frac{1}{3}(1+2\lambda)\hbar(W^{(1/2)}DW^{(1/2)}) \\
 &\quad - \frac{2}{3}\lambda(1+2\lambda)\hbar(W^{(1/2)}\partial W^{(1/2)}) \\
 &\quad - \frac{2}{3}\lambda(1-\lambda)\hbar(DW^{(1/2)}DW^{(1/2)}) \\
 &\quad + \frac{2}{3}\lambda(1-\lambda)\hbar^{3/2}\partial DW^{(1/2)}, \\
 (W^{(1/2)}W^{(3/2)}) &= \frac{1}{2}(W^{(1/2)}DW^{(1)}) - 2\lambda\sqrt{\hbar}(W^{(1/2)}\partial W^{(1/2)}). \tag{106}
 \end{aligned}$$

Of course, one can also extend the dependent identity (101) to the quantum level where it reads as

$$\begin{aligned}
 (W^{(1)}DW^{(1/2)}) - (W^{(1/2)}DW^{(1)}) &= \sqrt{\hbar}DW^{(3/2)} - 2\lambda\sqrt{\hbar}(DW^{(1/2)}DW^{(1/2)}) \\
 &\quad - 2\lambda\sqrt{\hbar}(W^{(1/2)}\partial W^{(1/2)}) \\
 &\quad + \frac{1}{2}\sqrt{\hbar}\partial W^{(1)} + 2\lambda\hbar\partial DW^{(1/2)}. \tag{107}
 \end{aligned}$$

We see that for  $\hbar \neq 0$  the first two identities in (106) can be used to solve for  $W^{(3/2)}$  and  $W^{(2)}$  in terms of (products of) lower-spin currents. We therefore conclude that at the quantum level there exists a telescoping procedure. It seems very suggestive that this telescoping mechanism extends to all other higher-spin generators with  $s \geq 2$  as well, but we have not proven this.

The above identities can be used to obtain representations of nonlinear algebras. Such a construction was performed for the bosonic  $W_N$  algebras in Ref. 63 and for the  $N = 1$  super- $W_2$  algebra in Ref. 64. The idea is to reinterpret some of the generators of the  $N = 2$  super- $W_\infty(\lambda)$  algebra as composite operators instead of independent generators. This is done by applying decomposition rules of the type given above to the right-hand side of the OPE expansions corresponding to the super- $W_\infty(\lambda)$  algebra. A linear algebra with an infinite number of generators is thus truncated to a nonlinear algebra with a finite number of generators.

From now we restrict ourselves to nonlinear algebras not containing an  $s = 1/2$  generator. We have seen in Sec. 5 that the  $s = 1/2$  generator is a quasi-primary generator that, without redefining the  $N = 2$  super-*Virasoro* generators, cannot be made primary. We will postpone a study of nonlinear algebras involving a spin-1/2 generator to future work. Given the above restriction we are forced to work either in the  $\lambda = 0$  or in the  $\lambda = 1/2$  basis. Only for these two choices of  $\lambda$  can the  $s = 1/2$  generator be truncated away consistently from the  $N = 2$  super- $W_\infty(\lambda)$  algebra. In Sec. 5 we have seen that at the level of Poisson brackets or single

contractions the two choices of  $\lambda$  are equivalent. This equivalence ceases to be true when multiple contractions are taken into account. Indeed, in Sec. 5 we saw that the  $\lambda = 0$  ( $\lambda = 1/2$ ) basis provides us with a  $c = 3$  ( $c = -3$ ) representation of the  $N = 2$  super- $W_\infty$  algebra. We will see that in the case of the nonlinear algebras multiple contractions already occur in noncentral terms in the OPE's. Consequently, the  $\lambda = 0$  and  $\lambda = 1/2$  cases lead to inequivalent nonlinear algebras. In both cases we will discuss the first example beyond the  $N = 2$  super-Virasoro algebra, corresponding to an  $N = 2$  super- $W_3$  algebra.

We first consider, both for  $\lambda = 0$  as well as  $\lambda = 1/2$ , identities between the classical currents of the  $N = 2$  super- $w_\infty$  algebra. To discuss the case of the  $N = 2$  super- $W_3$  algebra, it suffices to consider all representation-dependent relations up to a total spin  $s = 7/2$ . The independent relations are given by

$$\begin{aligned} w^{(1)}w^{(1)} &= 0, & w^{(1)}w^{(3/2)} &= 0, \\ w^{(1)}w^{(2)} &= 0, & w^{(1)}w^{(5/2)} &= 0, \\ w^{(3/2)}w^{(2)} &= 0, & Dw^{(1)}w^{(2)} &= 0. \end{aligned} \quad (108)$$

Below we give the quantum extensions of the above classical identities for the cases  $\lambda = 1/2$  and  $\lambda = 0$  separately.

#### A. The $\lambda = 1/2$ case

It turns out that for  $\lambda = 1/2$  there are no representation-dependent quantum identities at spin  $s = 2$  and  $s = 5/2$  not involving an  $s = 1/2$  generator. The quantum extension of the independent identities at  $s = 3$  and  $s = 7/2$  are given by

$$\begin{aligned} \sqrt{\hbar}W^{(3)} &= \frac{2}{3}(W^{(1)}W^{(2)}) + \frac{4}{9}\sqrt{\hbar}(W^{(1)}DW^{(3/2)}) \\ &\quad + \frac{2}{3}\sqrt{\hbar}(W^{(3/2)}DW^{(1)}) - \frac{1}{5}\hbar DW^{(5/2)} - \frac{2}{15}\hbar^{3/2}\partial^2 W^{(1)}, \\ \sqrt{\hbar}W^{(7/2)} &= 2(W^{(3/2)}W^{(2)}) + \frac{4}{3}\sqrt{\hbar}(W^{(3/2)}DW^{(3/2)}) \\ &\quad + \frac{2}{5}\hbar\partial DW^{(2)} - \frac{1}{6}\hbar^{3/2}\partial^2 W^{(3/2)} \end{aligned} \quad (109)$$

and

$$\begin{aligned} \frac{2}{3}(W^{(1)}DW^{(2)}) - \frac{4}{3}(DW^{(1)}W^{(2)}) &= \frac{4}{9}\sqrt{\hbar}(W^{(1)}\partial W^{(3/2)}) \\ &\quad + \frac{2}{9}\sqrt{\hbar}(DW^{(1)}DW^{(3/2)}) \\ &\quad - \frac{2}{3}\sqrt{\hbar}(W^{(3/2)}\partial W^{(1)}) \\ &\quad + \hbar\partial W^{(5/2)} - \frac{1}{3}\hbar^{3/2}\partial^2 DW^{(1)}, \end{aligned}$$

$$\begin{aligned}
 -(W^{(1)}W^{(5/2)}) + \frac{8}{3}(W^{(3/2)}W^{(2)}) &= -\frac{1}{3}\sqrt{\hbar}(W^{(1)}\partial DW^{(1)}) \\
 &\quad -\frac{4}{9}\sqrt{\hbar}(W^{(3/2)}DW^{(3/2)}) \\
 &\quad +\frac{1}{3}\sqrt{\hbar}(DW^{(1)}\partial W^{(1)}) \\
 &\quad -\frac{2}{3}\hbar\partial DW^{(2)} + \frac{4}{9}\hbar^{3/2}\partial^2 W^{(3/2)}. \quad (110)
 \end{aligned}$$

Furthermore, there is one dependent identity at spin  $s = 7/2$ :

$$\begin{aligned}
 \frac{2}{3}(W^{(1)}DW^{(2)}) + \frac{2}{3}(DW^{(1)}W^{(2)}) &= \sqrt{\hbar}DW^{(3)} - \frac{10}{9}\sqrt{\hbar}(DW^{(1)}DW^{(3/2)}) \\
 &\quad + \frac{4}{9}\sqrt{\hbar}(W^{(1)}\partial W^{(3/2)}) - \frac{2}{3}\sqrt{\hbar}(W^{(3/2)}\partial W^{(1)}) \\
 &\quad - \frac{1}{5}\hbar\partial W^{(5/2)} - \frac{1}{30}\hbar^{3/2}\partial^2 DW^{(1)}. \quad (111)
 \end{aligned}$$

### B. The $\lambda = 0$ case

We find for  $\lambda = 0$  the following independent quantum identities:

$$\begin{aligned}
 \sqrt{\hbar}W^{(2)} &= \frac{1}{2}(W^{(1)}W^{(1)}) - \frac{1}{3}\hbar DW^{(3/2)}, \\
 \sqrt{\hbar}W^{(5/2)} &= (W^{(1)}W^{(3/2)}) - \frac{1}{6}\hbar\partial DW^{(1)}, \\
 \sqrt{\hbar}W^{(3)} &= \frac{2}{3}(W^{(1)}W^{(2)}) + \frac{2}{9}\sqrt{\hbar}(W^{(1)}DW^{(3/2)}) - \frac{2}{5}\hbar DW^{(5/2)}, \\
 \sqrt{\hbar}W^{(7/2)} &= 2(W^{(3/2)}W^{(2)}) + \sqrt{\hbar}(W^{(1)}\partial DW^{(1)}) \\
 &\quad + \frac{2}{3}\sqrt{\hbar}(W^{(3/2)}DW^{(3/2)}) + \frac{1}{2}\sqrt{\hbar}(\partial W^{(1)}DW^{(1)}) \\
 &\quad - \frac{2}{5}\hbar\partial DW^{(2)} - \frac{2}{3}\hbar^{3/2}\partial^2 W^{(3/2)}
 \end{aligned} \quad (112)$$

and

$$\begin{aligned}
 (W^{(3/2)}W^{(2)}) - \frac{1}{2}(W^{(1)}W^{(5/2)}) &= -\frac{5}{12}\sqrt{\hbar}(W^{(1)}\partial DW^{(1)}) \\
 &\quad -\frac{1}{3}\sqrt{\hbar}(W^{(3/2)}DW^{(3/2)}) \\
 &\quad +\frac{1}{4}\hbar^{3/2}\partial^2 W^{(3/2)}, \\
 (DW^{(1)}W^{(2)}) - \frac{1}{2}(W^{(1)}DW^{(2)}) &= -\frac{5}{3}\sqrt{\hbar}(W^{(1)}\partial W^{(3/2)}) \\
 &\quad -\frac{1}{3}\sqrt{\hbar}(DW^{(1)}DW^{(3/2)}) \\
 &\quad +\frac{1}{4}\hbar^{3/2}\partial^2 DW^{(1)}. \quad (113)
 \end{aligned}$$



In addition, there are several more dependent identities which we do not give here.

We see that for  $\hbar \neq 0$  and  $\lambda = 1/2$  the identities (109) can be used to solve  $W^{(3)}$  and  $W^{(7/2)}$  in terms of product of lower-spin currents. On the other hand, for  $\hbar \neq 0$  and  $\lambda = 0$  the identities (112) can be applied to solve  $W^{(s)}$  with  $2 \leq s \leq 7/2$ . We therefore conclude that, both for  $\lambda = 0$  as well as  $\lambda = 1/2$  there exists a quantum telescoping procedure, which does not introduce an  $s = 1/2$  generator.

In the case  $\lambda = 1/2$  the construction leads to a  $c = -3$  representation of an  $N = 2$  super- $W_3$  algebra with as independent currents the set  $\{W^{(1)}, W^{(3/2)}, W^{(2)}, W^{(5/2)}\}$ . From the OPE expansions given in the appendix we see that the operator products of these currents only give rise to  $W^{(3)}$  and  $W^{(7/2)}$  as new operators. We use the identities (109) to solve these operators as composite expressions in terms of the independent generators of the  $N = 2$  super- $W_3$  algebra. The resulting OPE expansions for the currents can be easily derived by substituting  $\lambda = 1/2$  in the OPE expansions given in the appendix. In the resulting expressions it is understood that everywhere  $W^{(3)}$  and  $W^{(7/2)}$  are given by the nonlinear expressions (109). We should note that the  $\{W^{(1)}, W^{(3/2)}\}$  currents form a  $c = -3$  representation of a  $N = 2$  super-*Virasoro* algebra.

A similar construction can be performed for the  $\lambda = 0$  case. Like in the  $\lambda = 1/2$  case, we take as independent currents the set  $\{W^{(s)}\}$  with  $1 \leq s \leq 5/2$ . We now use the OPE expansions given in the appendix for  $\lambda = 0$ . Again, these OPE's only give rise to  $W^{(3)}$  and  $W^{(7/2)}$  as independent currents. Next, we use the third and fourth identity in (112) to solve these operators. The  $\lambda = 0$  construction thus leads to a  $c = 3$  representation of another  $N = 2$  super- $W_3$  algebra. To distinguish it from the  $\lambda = 1/2$  case we will call this algebra  $N = 2$  super- $W'_3$ .

A few remarks are in order. First of all, we should note that there is an arbitrariness in the way we decide to write the  $W^{(3)}$  and  $W^{(7/2)}$  currents as composite operators. This is due to the fact that there are more representation-dependent relations at spin-3 and spin-7/2, than the ones which are used to solve the  $W^{(3)}$  and  $W^{(7/2)}$  operators (see Eqs. (110), (111) for  $\lambda = 1/2$  and Eq. (113) for  $\lambda = 0$ ). One may always add to a given decomposition these identities multiplied by an arbitrary coefficient. The reason that this arbitrariness occurs is that we are working in the context of a special two-scalar superfield realization at  $c = -3$ ,  $c = +3$  for  $\lambda = 1/2$ ,  $\lambda = 0$ , respectively. Therefore, our result may only be viewed as the  $c = -3$  ( $c = +3$ ) realization of a  $N = 2$  super- $W_3$  ( $N = 2$  super- $W'_3$ ) algebra at arbitrary  $c$  modulo the special  $c = -3$  ( $c = +3$ ) identities mentioned above. To fix the arbitrariness one should use a representation based upon  $2i$  ( $i \geq 2$ ) scalar superfields

$$\{\phi_i, \bar{\phi}_i\}, \quad i \geq 2. \quad (114)$$

These fields should provide a representation of the algebra at *arbitrary* value of the central charge. For instance, the  $i = 2$  representation of the  $N = 2$  super- $W_3$  algebra was given recently in Refs. 19 and 20. More recently, using a technique of<sup>65</sup>,

a representation of the  $N = 2$  super- $W_3$  algebra for arbitrary values of  $i$  has been given.<sup>66</sup>

Secondly, in the  $i = 1$  representation the supercurrents  $W^{(2)}$  and  $W^{(5/2)}$  are not primary with respect to the  $N = 1$  super-*Virasoro* algebra generated by  $W^{(3/2)}$  (they are quasi-primary though). The question now is whether we can make these currents primary by an appropriate (nonlinear) redefinition. In Eqs. (81), (82) it is shown that such a redefinition is possible for arbitrary value of  $\lambda$ . For  $\lambda = 1/2$  the supercurrents  $\{W^{(2)}, W^{(5/2)}\}$  can be made  $N = 2$  primary with respect to the  $N = 2$  super-*Virasoro* algebra generated by  $\{W^{(1)}, W^{(3/2)}\}$ . More explicitly, for  $\lambda = 1/2$  the redefinitions (81), (82) can be written in terms of the generators of the  $N = 2$  super- $W_3$  algebra (i.e. without  $W^{(1/2)}$ ) as

$$\begin{aligned} W^{(2)'} &= 4W^{(2)} - (W^{(1)}W^{(1)}) + \frac{2}{3}DW^{(3/2)}, \\ W^{(5/2)'} &= 3W^{(5/2)} - 2(W^{(1)}W^{(3/2)}). \end{aligned} \quad (115)$$

It turns out that for  $\lambda = 0$  the redefinitions (81) involve the  $s = 1/2$  generator. Therefore, for  $\lambda = 0$ , the algebra contains quasi-primary generators that cannot be made primary by a suitable redefinition. This shows that the  $\lambda = 0$  and  $\lambda = 1/2$  cases lead two inequivalent algebras.

Finally, we note that in general it might happen that the particular value of the central charge we are working with corresponds to a singularity in the expressions for the super- $W$  algebra at arbitrary  $c$ .<sup>w</sup> For instance, in the  $N = 2$  super- $W_3$  algebra of Ref. 18 a factor of  $\frac{1}{c+3}$  occurs. Therefore, for  $c = -3$  some of the expressions become singular.<sup>x</sup> Since the results of Ref. 18 are given in a primary basis it is difficult to compare. We expect that in the  $c = -3$  representation we are using all the expressions multiplying a  $\frac{1}{c+3}$  factor are zero identically. Indeed we note that there are nontrivial nonlinear identities at spin 3 and  $7/2$  [see Eqs. (110) and (111)]. It would be interesting to verify whether this is indeed the case.

## 8. Conclusions

In this paper we have quantized the classical gauge theory of  $N = 2$   $w_\infty$ -supergravity. We have used a representation in terms of two scalar superfields corresponding to a two-dimensional target space with Minkowskian signature. Like in the bosonic case we find that the underlying classical algebra deforms into a corresponding quantum algebra, thereby removing all matter-dependent anomalies. The universal anomalies can be canceled as well by choosing an appropriate matter system.

We have shown how our results can be used to obtain a two-scalar superfield realization of quantum  $W_3$ -supergravity. We gave the explicit answer for the case of quantum  $N = 2$   $W_3$ -supergravity at  $c = -3$  and quantum  $N = 2$  super- $W'_3$ -supergravity at  $c = +3$ . In the first case all generators of the underlying algebra

<sup>w</sup>We thank C. Pope and K. Schoutens for a discussion on the issue raised in this paragraph.

<sup>x</sup>The singularity at  $c = -3$  is also discussed in Ref. 18.

can be made primary and, although we have not explicitly checked this, we expect that this case provides a  $c = -3$  realization of the  $N = 2$  super- $W_3$  algebra given in Ref. 18. In the latter case the underlying algebra contains quasi-primary generators that cannot be made primary by some suitable redefinition. It would be interesting to investigate whether or not this case corresponds to the  $c = +3$  realization of a  $N = 2$  super- $W'_3$  algebra at arbitrary values of  $c$ . In this context it is interesting to note that the occurrence of an  $N = 2$  super- $W$  algebra with quasi-primary generators is also mentioned in Ref. 15.

It would be interesting to extend our work to the general case of quantum  $N = 2$   $W_N$ -supergravity theories. We expect that the  $N = 3$  decomposition rules described in this paper should also work for  $N > 3$ . One question that arises here is whether the higher-spin generators can be made  $N = 2$  primary for the same value of the parameter  $\lambda$ .

An unusual feature we encountered is that the telescoping procedure needed to perform the truncation to super- $W_N$  only exists at the quantum level but not at the classical level. We note that the situation is different in the bosonic case. The difference can be understood from the following dimension counting argument. Consider the schematic form of the currents in the bosonic and supersymmetric case:

$$\begin{aligned} w^{(s)} &\sim (\partial\phi)^s && \text{bosonic,} \\ w^{(s)} &\sim (\partial\phi)^{s-1} D\phi D\bar{\phi} && \text{supersymmetric.} \end{aligned} \tag{116}$$

From these expressions we see that in the bosonic case  $w^{(s)}$  and  $w^{(t)}w^{(u)}$  with  $s = t + u$  have the same dimensions (we remind that the dimension of  $\phi$  is  $\sqrt{\hbar}$ ). However, in the supersymmetric case  $w^{(s)}$  has dimension  $(s+1)\sqrt{\hbar}$  whereas  $w^{(t)}w^{(u)}$  has dimension  $(s+2)\sqrt{\hbar}$ . Therefore, in the supersymmetric case one can only have identities of the form  $w^{(s)} = 1/\sqrt{\hbar}w^{(t)}w^{(u)}$  which only makes sense at the quantum level. Due to the  $1/\sqrt{\hbar}$  factors, using the two-scalar superfield realization of quantum  $W_N$ -supergravity, one cannot take the classical limit to obtain a realization of a classical  $w_N$ -supergravity theory. Only for  $N = \infty$  a classical supergravity theory in terms of commuting superfields (allowing a superstring coordinate interpretation) can be defined. This raises the issue of how to define the classical limit of a quantum  $W_N$  superstring theory.

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## Appendix A. Currents and OPE's

In this appendix we give the explicit forms of all currents  $W_\lambda^{(s)}$  with  $1/2 \leq s \leq 5/2$  and the (singular part of the) operator product expansions between them. These

expressions can be derived by a straightforward application of the general formulae given in Sec. 5.

The first few supercurrents take the form

$$\begin{aligned}
 W_\lambda^{(1/2)} &= BC, \\
 W_\lambda^{(1)} &= (1 - 2\lambda)(DB)C - 2\lambda B(DC), \\
 W_\lambda^{(3/2)} &= \frac{1}{2}(1 - 2\lambda)(\partial B)C - \frac{1}{2}(DB)(DC) - \lambda B(\partial C), \\
 W_\lambda^{(2)} &= \frac{1}{3}(1 - 2\lambda)(1 - \lambda)(\partial DB)C - \frac{1}{3}(1 + 2\lambda)(1 - \lambda)(\partial B)(DC) \\
 &\quad - \frac{1}{3}(1 + 2\lambda)(1 - \lambda)(DB)(\partial C) + \frac{1}{3}\lambda(2\lambda + 1)B(\partial DC), \\
 W_\lambda^{(5/2)} &= \frac{1}{6}(1 - 2\lambda)(1 - \lambda)(\partial^2 B)C - \frac{1}{3}(1 - \lambda)(\partial DB)(DC) \\
 &\quad - \frac{1}{3}(1 + 2\lambda)(1 - \lambda)(\partial B)(\partial C) + \frac{1}{6}(1 + 2\lambda)(DB)(\partial DC) \\
 &\quad + \frac{1}{6}\lambda(1 + 2\lambda)B(\partial^2 C) \\
 &\vdots
 \end{aligned} \tag{117}$$

The operator product expansions between these currents are given by (we set  $\hbar = 1$ )

$$\begin{aligned}
 W_\lambda^{(1/2)}(1)W_\lambda^{(1/2)}(2) &\sim \text{regular}, \\
 W_\lambda^{(1/2)}(1)W_\lambda^{(1)}(2) &\sim \frac{W_\lambda^{(1/2)}}{z_{12}} + \frac{\theta_{12}}{z_{12}^2} + \text{regular}, \\
 W_\lambda^{(3/2)}(1)W_\lambda^{(1/2)}(2) &\sim \left\{ \frac{1}{2} \frac{\theta_{12} W_\lambda^{(1/2)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 W_\lambda^{(1/2)}}{z_{12}} + \frac{\theta_{12} \partial_2 W_\lambda^{(1/2)}}{z_{12}} \right\} \\
 &\quad - \frac{1}{2} \frac{1}{z_{12}^2} + \text{regular}, \\
 W_\lambda^{(1/2)}(1)W_\lambda^{(2)}(2) &\sim \frac{W_\lambda^{(3/2)}}{z_{12}} + \left\{ \frac{\theta_{12} W_\lambda^{(1)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 W_\lambda^{(1)}}{z_{12}} \right\} \\
 &\quad - \frac{4}{3} \left( \lambda - \frac{1}{4} \right) \left\{ \frac{W_\lambda^{(1/2)}}{z_{12}^2} - \frac{\theta_{12} D_2 W_\lambda^{(1/2)}}{z_{12}^2} - \frac{\partial_2 W_\lambda^{(1/2)}}{z_{12}} \right\} \\
 &\quad - \frac{4}{3} \left( \lambda - \frac{1}{4} \right) \frac{\theta_{12}}{z_{12}^3} + \text{regular},
 \end{aligned}$$

$$\begin{aligned}
W_\lambda^{(1/2)}(1)W_\lambda^{(5/2)}(2) \sim & \frac{1}{2} \left\{ 3 \frac{\theta_{12}W_\lambda^{(3/2)}}{z_{12}^2} - \frac{D_2W_\lambda^{(3/2)}}{z_{12}} \right\} \\
& + \frac{1}{4} \left\{ 2 \frac{W_\lambda^{(1)}}{z_{12}^2} - \frac{\theta_{12}D_2W_\lambda^{(1)}}{z_{12}^2} - \frac{\partial_2W_\lambda^{(1)}}{z_{12}} \right\} \\
& - \frac{1}{3} \left( \lambda - \frac{1}{4} \right) \left\{ 4 \frac{\theta_{12}W_\lambda^{(1/2)}}{z_{12}^3} - 4 \frac{D_2W_\lambda^{(1/2)}}{z_{12}^2} \right. \\
& \left. - 4 \frac{\theta_{12}\partial_2W_\lambda^{(1/2)}}{z_{12}^2} + 2 \frac{\partial_2D_2W_\lambda^{(1/2)}}{z_{12}^3} \right\} \\
& - \frac{2}{3} \left( \lambda - \frac{1}{4} \right) \frac{1}{z_{12}^3} + \text{regular},
\end{aligned}$$

$$W_\lambda^{(1)}(1)W_\lambda^{(1)}(2) \sim -2 \frac{\theta_{12}W_\lambda^{(3/2)}}{z_{12}} - 4 \frac{\lambda - \frac{1}{4}}{z_{12}^2} + \text{regular},$$

$$W_\lambda^{(3/2)}(1)W_\lambda^{(1)}(2) \sim \left\{ \frac{\theta_{12}W_\lambda^{(1)}}{z_{12}^2} - \frac{1}{2} \frac{D_2W_\lambda^{(1)}}{z_{12}} + \frac{\theta_{12}\partial_2W_\lambda^{(1)}}{z_{12}} \right\} + \text{regular},$$

$$\begin{aligned}
W_\lambda^{(1)}(1)W_\lambda^{(2)}(2) \sim & -2 \frac{\theta_{12}W_\lambda^{(5/2)}}{z_{12}} - \frac{8}{3} \left( \lambda - \frac{1}{4} \right) \frac{W_\lambda^{(1)}}{z_{12}^2} \\
& + 4\lambda \left( \lambda - \frac{1}{2} \right) \left\{ \frac{\theta_{12}W_\lambda^{(1/2)}}{z_{12}^3} - \frac{D_2W_\lambda^{(1/2)}}{z_{12}^2} \right\} \\
& + 4 \frac{\lambda(\lambda - \frac{1}{2})}{z_{12}^3} + \text{regular},
\end{aligned}$$

$$\begin{aligned}
W_\lambda^{(1)}(1)W_\lambda^{(5/2)}(2) \sim & -\frac{1}{2} \left\{ 4 \frac{\theta_{12}W_\lambda^{(2)}}{z_{12}^2} - \frac{D_2W_\lambda^{(2)}}{z_{12}} + \frac{\theta_{12}\partial_2W_\lambda^{(2)}}{z_{12}} \right\} \\
& - \frac{8}{3} \left( \lambda - \frac{1}{4} \right) \frac{W_\lambda^{(3/2)}}{z_{12}^2} \\
& + 2\lambda \left( \lambda - \frac{1}{2} \right) \left\{ \frac{W_\lambda^{(1/2)}}{z_{12}^3} - \frac{\theta_{12}D_2W_\lambda^{(1/2)}}{z_{12}^3} - \frac{\partial_2W_\lambda^{(1/2)}}{z_{12}^2} \right\} \\
& + 2\lambda \left( \lambda - \frac{1}{2} \right) \frac{\theta_{12}}{z_{12}^4} + \text{regular},
\end{aligned}$$

$$\begin{aligned}
W_\lambda^{(3/2)}(1)W_\lambda^{(3/2)}(2) \sim & \left\{ \frac{3}{2} \frac{\theta_{12}W_\lambda^{(3/2)}}{z_{12}^2} - \frac{1}{2} \frac{D_2W_\lambda^{(3/2)}}{z_{12}} + \frac{\theta_{12}\partial_2W_\lambda^{(3/2)}}{z_{12}} \right\} \\
& + 2 \frac{\lambda - \frac{1}{4}}{z_{12}^3} + \text{regular},
\end{aligned}$$

$$\begin{aligned}
 W_\lambda^{(3/2)}(1)W_\lambda^{(2)}(2) &\sim \left\{ 2\frac{\theta_{12}W_\lambda^{(2)}}{z_{12}^2} - \frac{1}{2}\frac{D_2W_\lambda^{(2)}}{z_{12}} + \frac{\theta_{12}\partial_2W_\lambda^{(2)}}{z_{12}} \right\} \\
 &\quad - 2\lambda\left(\lambda - \frac{1}{2}\right)\frac{W_\lambda^{(1/2)}}{z_{12}^3} - 2\lambda\left(\lambda - \frac{1}{2}\right)\frac{\theta_{12}}{z_{12}^4} + \text{regular}, \\
 W_\lambda^{(3/2)}(1)W_\lambda^{(5/2)}(2) &\sim \left\{ \frac{5}{2}\frac{\theta_{12}W_\lambda^{(5/2)}}{z_{12}^2} - \frac{1}{2}\frac{D_2W_\lambda^{(5/2)}}{z_{12}} + \frac{\theta_{12}\partial_2W_\lambda^{(5/2)}}{z_{12}} \right\} \\
 &\quad + \frac{4}{3}\left(\lambda - \frac{1}{4}\right)\frac{W_\lambda^{(1)}}{z_{12}^3} \\
 &\quad - 3\lambda\left(\lambda - \frac{1}{2}\right)\left\{ \frac{\theta_{12}W_\lambda^{(1/2)}}{z_{12}^4} - \frac{D_2W_\lambda^{(1/2)}}{z_{12}^3} \right\} \\
 &\quad - 3\lambda\left(\lambda - \frac{1}{2}\right)\frac{1}{z_{12}^4} + \text{regular}, \\
 W_\lambda^{(2)}(1)W_\lambda^{(2)}(2) &\sim -2\frac{\theta_{12}W_\lambda^{(7/2)}}{z_{12}} \\
 &\quad - \frac{8}{3}\left(\lambda - \frac{1}{4}\right)\left\{ \frac{W_\lambda^{(2)}}{z_{12}^2} + \frac{1}{2}\frac{\theta_{12}D_2W_\lambda^{(2)}}{z_{12}^2} \right. \\
 &\quad \left. + \frac{1}{2}\frac{\partial_2W_\lambda^{(2)}}{z_{12}} + \frac{3}{10}\frac{\theta_{12}\partial_2D_2W_\lambda^{(2)}}{z_{12}} \right\} \\
 &\quad + \frac{4}{3}(\lambda - 1)(2\lambda + 1)\left\{ \frac{\theta_{12}W_\lambda^{(3/2)}}{z_{12}^3} - \frac{1}{3}\frac{D_2W_\lambda^{(3/2)}}{z_{12}^2} \right. \\
 &\quad \left. + \frac{2}{3}\frac{\theta_{12}\partial_2W_\lambda^{(3/2)}}{z_{12}^2} - \frac{1}{6}\frac{\partial_2D_2W_\lambda^{(3/2)}}{z_{12}} + \frac{1}{4}\frac{\theta_{12}\partial_2^2W_\lambda^{(3/2)}}{z_{12}} \right\} \\
 &\quad - \frac{4}{3}\left(\lambda - \frac{1}{4}\right)\frac{1 + 2\lambda - 4\lambda^2}{z_{12}^4} + \text{regular}, \\
 W_\lambda^{(2)}(1)W_\lambda^{(5/2)}(2) &\sim \left\{ -3\frac{\theta_{12}W_\lambda^{(3)}}{z_{12}^2} + \frac{1}{2}\frac{D_2W_\lambda^{(3)}}{z_{12}} - \frac{3}{2}\frac{\theta_{12}\partial_2W_\lambda^{(3)}}{z_{12}} \right\} \\
 &\quad - \frac{4}{3}\left(\lambda - \frac{1}{4}\right)\left\{ \frac{W_\lambda^{(5/2)}}{z_{12}^2} + \frac{2}{5}\frac{\theta_{12}D_2W_\lambda^{(5/2)}}{z_{12}^2} \right. \\
 &\quad \left. + \frac{2}{5}\frac{\partial_2W_\lambda^{(5/2)}}{z_{12}} + \frac{1}{5}\frac{\theta_{12}\partial_2D_2W_\lambda^{(5/2)}}{z_{12}} \right\} \\
 &\quad + \frac{1}{6}(\lambda - 1)(2\lambda + 1)\left\{ 4\frac{\theta_{12}W_\lambda^{(1)}}{z_{12}^4} - 2\frac{D_2W_\lambda^{(1)}}{z_{12}^3} \right. \\
 &\quad \left. + 2\frac{\theta_{12}\partial_2W_\lambda^{(1)}}{z_{12}^3} - \frac{2}{3}\frac{\partial_2D_2W_\lambda^{(1)}}{z_{12}^2} + \frac{2}{3}\frac{\theta_{12}\partial_2^2W_\lambda^{(1)}}{z_{12}^2} \right. \\
 &\quad \left. - \frac{1}{6}\frac{\partial_2^2D_2W_\lambda^{(1)}}{z_{12}} + \frac{1}{6}\frac{\theta_{12}\partial_2^3W_\lambda^{(1)}}{z_{12}} \right\} + \text{regular},
 \end{aligned}$$

$$\begin{aligned}
W_\lambda^{(5/2)}(1)W_\lambda^{(5/2)}(2) \sim & \left\{ \frac{7}{2} \frac{\theta_{12} W_\lambda^{(7/2)}}{z_{12}^2} - \frac{1}{2} \frac{D_2 W_\lambda^{(7/2)}}{z_{12}} + 2 \frac{\theta_{12} \partial_2 W_\lambda^{(7/2)}}{z_{12}} \right\} \\
& + \left( \lambda - \frac{1}{4} \right) \left\{ \frac{4}{3} \frac{W_\lambda^{(2)}}{z_{12}^3} + \frac{2}{3} \frac{\theta_{12} D_2 W_\lambda^{(2)}}{z_{12}^3} + \frac{2}{3} \frac{\partial_2 W_\lambda^{(2)}}{z_{12}^2} \right. \\
& + \frac{2}{5} \frac{\theta_{12} \partial_2 D_2 W_\lambda^{(2)}}{z_{12}^2} + \frac{1}{5} \frac{\partial_2^2 W_\lambda^{(2)}}{z_{12}} + \frac{2}{15} \frac{\theta_{12} \partial_2^2 D_2 W_\lambda^{(2)}}{z_{12}} \left. \right\} \\
& + \frac{1}{6} (\lambda - 1)(2\lambda + 1) \left\{ -10 \frac{\theta_{12} W_\lambda^{(3/2)}}{z_{12}^4} + \frac{10}{3} \frac{D_2 W_\lambda^{(3/2)}}{z_{12}^3} \right. \\
& - \frac{20}{3} \frac{\theta_{12} \partial_2 W_\lambda^{(3/2)}}{z_{12}^3} + \frac{5}{3} \frac{\partial_2 D_2 W_\lambda^{(3/2)}}{z_{12}^2} - \frac{5}{2} \frac{\theta_{12} \partial_2^2 W_\lambda^{(3/2)}}{z_{12}^2} \\
& + \frac{1}{2} \frac{\partial_2^2 D_2 W_\lambda^{(3/2)}}{z_{12}} - \frac{2}{3} \frac{\theta_{12} \partial_2^3 W_\lambda^{(3/2)}}{z_{12}} \left. \right\} \\
& + \frac{4}{3} \left( \lambda - \frac{1}{4} \right) \frac{1 + 2\lambda - 4\lambda^2}{z_{12}^5} + \text{regular}. \tag{118}
\end{aligned}$$

We have only given one order of the currents in each OPE. The expression for the other order can be easily derived by using the identity  $W_\lambda^{(s)}(1)W_\lambda^{(t)}(2) = (-)^{|2s|_2|2t|_2} W_\lambda^{(t)}(2)W_\lambda^{(s)}(1)$  and the super-Taylor expansion rule.

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